１次元非線形 KLEIN-GORDON 方程式系の散乱問題

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We consider the Cauchy problem for the system of semi-linear Klein-Gordon equations

$$
\begin{cases} \left(\partial_t^2 - \partial_x^2 + m_j^2\right) u_j = \mathcal{N}_j\left(\partial u\right), & (t, x) \in \mathbf{R} \times \mathbf{R}, \\ u_j\left(0, x\right) = \overset{\circ}{u}{}_j^{(1)}\left(x\right), \ \partial_t u_j\left(0, x\right) = \overset{\circ}{u}{}_j^{(2)}\left(x\right), & x \in \mathbf{R}, \end{cases} \tag{0.1}
$$

where $j = 1, ..., l$, $m_j > 0$, the partial derivative $\partial = (\partial_t, \partial_x)$ and $u = (u_1, ..., u_l)$. We assume that $\mathcal{N}_j\left(\partial u\right)$ are quadratic nonlinearities. Our purpose is to prove global existence of small solutions and to consider a scattering problem for equation (0.1) under the strong null condition on the nonlinearities $\mathcal{N}_j$ introduced by [3] which is written as

$$
\mathcal{N}_j\left(\partial u\right) = \sum_{p,q=1}^{l} A_{jpq}\left(\left(\partial_t u_p\right)\partial_x u_q - \left(\partial_x u_p\right)\partial_t u_q\right) \tag{0.2}
$$

where $A_{jpq} \in \mathbf{C}$. Condition (0.2) implies an additional time decay of order t^{-1} through the operator $\mathcal{Z} = x\partial_t + t\partial_x$ since

$$
\left(\left(\partial_t u_p\right)\partial_x u_q - \left(\partial_x u_p\right)\partial_t u_q\right) = \frac{1}{t}\left(\left(\partial_t u_p\right)\mathcal{Z}u_q - \left(\mathcal{Z}u_p\right)\partial_t u_q\right).
$$

However we encounter the derivative loss with respect to the operator $\mathcal{Z}$. To overcome the derivative loss we use an analytic function space including the operator $\mathcal{Z}$. The operator $\mathcal{Z}$ was used previously by Klainerman [7] to prove global existence theorem for the nonlinear Klein-Gordon equations with quadratic nonlinearities in three space dimensions (see also papers [1], [3], [4], [6], [8], [9]). Global existence of small solutions to cubic nonlinear Klein-Gordon equations in one space dimension was studied extensively. Non resonance cubic nonlinearities were studied in [6] for a single equation and in [9] for a system of equations with different masses. In [2], [5], [10], resonance cubic nonlinearities were treated. For the case of quadratic nonlinearities there are few results. In paper [8], it was studied an almost global existence of small solutions to semi-linear Klein-Gordon equations for a single case. As far as we know there are no global results for a system of nonlinear Klein-Gordon equations in the case of quadratic nonlinearities.

In order to explain the analytic function space used in this paper we now state the notations. Let $\mathbf{L}^p$ be the usual Lebesgue space with the norm $\|\phi\|_{\mathbf{L}^p} = \left(\int_{\mathbf{R}} |\phi(x)|^p dx\right)^{\frac{1}{p}}$ if $1 \le p < \infty$ and $\|\phi\|_{\mathbf{L}^\infty} = \sup_{x\in\mathbf{R}} |\phi(x)|$ if $p = \infty$. Sobolev space is

$$
\mathbf{H}_p^m = \left\{ \phi \in \mathbf{L}^p : \|\phi\|_{\mathbf{H}_p^m} \equiv \sum_{j=0}^{m} \left\|\partial_x^j \phi\right\|_{\mathbf{L}^p} < \infty \right\},
$$

Key words and phrases. Klein-Gordon 方程式系, 散乱問題, １次元.

where $m \in \mathbf{N}$, $1 \le p \le \infty$. We also write $\mathbf{H}^m = \mathbf{H}_2^m$ for simplicity. We let

$$\mathcal{Q} = (\partial_t, \partial_x, \mathcal{Z}), \ \mathcal{P} = (x, \partial_x, \partial_t, \mathcal{Y}, \mathcal{Z}), \ \mathcal{Y} = x\partial_x + t\partial_t, \ \mathcal{Z} = x\partial_t + t\partial_x$$

and

$$\mathbf{X}_n = \left\{ \phi \in \mathbf{L}^2 : \|\phi\|_{\mathbf{X}_n} = \sum_{|\alpha| \le n} \|\mathcal{Q}^\alpha \phi\|_{\mathbf{L}^2} < \infty \right\}, n \in \mathbf{N}.$$

We use the same notations for vector-functions, for example we write $\|f\|_{\mathbf{H}_p^m} = \sum_{j=1}^{l} \|f_j\|_{\mathbf{H}_p^m}$ for a vector $f = (f_1, ..., f_l)$. Different positive constants we denote by the same letter C. We define an analytic function space as follows:

$$\mathbf{G}^{\mathbf{A}}(\mathcal{A}; \mathbf{X}) = \left\{ f \in \mathbf{X}; \|f\|_{\mathbf{G}^{\mathbf{A}}(\mathcal{A};\mathbf{X})} = \sum_{\alpha \ge 0} \frac{A^\alpha}{\alpha!} \|\mathcal{A}^\alpha f\|_{\mathbf{X}} < \infty \right\},$$

where $A = (A_1, ..., A_N)$, $A_j > 0$, $\mathcal{A} = (\mathcal{A}_1, ..., \mathcal{A}_N)$, $\alpha! = \prod_{j=1}^N \alpha_j!$, $|\alpha| = \sum_{j=1}^N \alpha_j$, $\alpha \ge 0$ means that $\alpha_j \ge 0$ for $1 \le j \le N$, and $\mathbf{X}$ is a Banach space. It is easy to see that

$$\mathbf{G}^{A_1 \dots A_N}(\mathcal{A}_1, \mathcal{A}_2, ..., \mathcal{A}_N; \mathbf{X}) = \mathbf{G}^{A_2 \dots A_N}\left(\mathcal{A}_2, ..., \mathcal{A}_N; \mathbf{G}^{A_1}(\mathcal{A}_1; \mathbf{X})\right).$$

Our basic analytic function space is $\mathbf{G}^{\mathbf{a}}\left(\partial_t, \partial_x, \mathcal{Z}; \mathbf{L}^2\right)$, $\mathbf{a} = (a_1, a_2, a_3)$. To prove a-priori estimate of solutions in the neighborhood of $t = 0$ in the class $\mathbf{G}^{\mathbf{a}}\left(\partial_t, \partial_x, \mathcal{Z}; \mathbf{L}^2\right)$ we need to show for some small T

$$\sup_{t \in [0,T]} \|u(t)\|_{\mathbf{G}^{\mathbf{a}}(\partial_t, \partial_x, x\partial_t; \mathbf{L}^2)} < \infty.$$

Since ∂_t is equivalent to $\sqrt{m_j^2 - \partial_x^2}$ in the linear case, so this estimate is naturally related with a-priori estimate

$$\sup_{t \in [0,T]} \|u(t)\|_{\mathbf{G}^{\mathbf{a}}(x, \partial_x, x\partial_x; \mathbf{L}^2)} < \infty.$$

First we state the local existence result. Denote $\mathcal{B} = (x, \partial_x, \mathcal{Y})$.

Theorem 0.1. *Assume that for some constant vector* $\mathbf{A} = (A_1, A_2, A_3)$ *with* $A_1, A_2 > 0, 0 < A_3 < 1$ *the norms*

$$\left\| \overset{\circ}{u}{}_j^{(1)} \right\|_{\mathbf{G}^{\mathbf{A}}(x, \partial_x, x\partial_x; \mathbf{H}^2)} + \left\| \overset{\circ}{u}{}_j^{(2)} \right\|_{\mathbf{G}^{\mathbf{A}}(x, \partial_x, x\partial_x; \mathbf{H}^1)} < \infty.$$

Then for some $T > 0$ *(which depends on the size of the initial data) there exists a unique solution of (0.1) which satisfies the estimates*

$$\sup_{0 \le t \le T} \left(\|u(t)\|_{\mathbf{G}^{\mathbf{A}}(\mathcal{B}; \mathbf{H}^2)} + \|\partial_t u(t)\|_{\mathbf{G}^{\mathbf{A}}(\mathcal{B}; \mathbf{H}^1)} \right) < \infty.$$

Moreover for some constant vector $\mathbf{a}$ *the solution satisfies the estimate*

$$\sup_{0 \le t \le T} \|u(t)\|_{\mathbf{G}^{\mathbf{a}}(\mathcal{P}; \mathbf{H}^2)} < \infty.$$

Remark 0.1. *Typical example of the initial function is given by* $\varepsilon \exp\left(-x^2\right)$ *which decays exponentially at infinity and has an analytic continuation on the strip and on the sector. Therefore* $\exp\left(-x^2\right) \in \mathbf{G}^{\mathbf{A}}\left(x, \partial_x, x\partial_x; \mathbf{H}^2\right)$.

We now state a global existence and asymptotics of solutions.

Theorem 0.2. *Assume that for some constant vector* $\mathbf{A} = (A_1, A_2, A_3)$ *with* $A_1, A_2 > 0, 0 < A_3 < 1$ *the norms*

$$\left\| \overset{\circ}{u}{}_j^{(1)} \right\|_{\mathbf{G}^{\mathbf{A}}(x,\partial_x,x\partial_x;\mathbf{H}^2)} + \left\| \overset{\circ}{u}{}_j^{(2)} \right\|_{\mathbf{G}^{\mathbf{A}}(x,\partial_x,x\partial_x;\mathbf{H}^1)} < \varepsilon$$

with some small $\varepsilon > 0$. *Furthermore suppose that the strong null condition (0.2) is fulfilled. Then the Cauchy problem (0.1) has a unique global solution* u *such that*

$$u_j \in \mathbf{C}\left([0,\infty); \mathbf{G}^{\mathbf{a}}\left(\mathcal{Q}; \mathbf{X}_5\right)\right)$$

and

$$\|u(t)\|_{\mathbf{G}^a(\partial_x;\mathbf{L}^\infty)} \leq C \langle t \rangle^{-\frac{1}{2}}$$

for all $t \geq 0$, *where* $\mathbf{a} = (a, a, a)$, $a > 0$ *is a small positive constant depending on* $\mathbf{A}, \varepsilon$. *Furthermore there exists a unique final state* $u_j^{+(1)}, u_j^{+(2)} \in \mathbf{G}^a\left(\partial_x; \mathbf{L}^2\right)$ *satisfying*

$$\left\| u_j(t) - \left(\cos\left(t\sqrt{m_j^2 - \partial_x^2} \right) u_j^{+(1)} + \frac{\sin\left(t\sqrt{m_j^2 - \partial_x^2}\right)}{\sqrt{m_j^2 - \partial_x^2}} u_j^{+(2)} \right) \right\|_{\mathbf{G}^a(\partial_x;\mathbf{L}^2)} \leq C\varepsilon^2 \langle t \rangle^{-\frac{1}{2}}$$

for all $t \geq 0$, $1 \leq j \leq l$.

Acknowledgment. This is a joint work with Pavel Naumkin.

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