# Global Existence and Uniqueness of Solutions to the Maxwell-Schrödinger Equations

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## §1. Introduction

The Maxwell-Schrödinger system (MS) in space dimension 3 describes the time evolution of a charged nonrelativistic quantum mechanical particle interacting with the (classical) electro-magnetic field it generates. We can state this system in usual vector notation as follows:

$$i\partial_t u = (-\Delta_A + \phi)u, \tag{1.1}$$

$$-\Delta\phi - \partial_t \operatorname{div} A = \rho, \tag{1.2}$$

$$\Box A + \nabla(\partial_t \phi + \operatorname{div} A) = J, \tag{1.3}$$

where $(u, \phi, A) : \boldsymbol{R}^{1+3} \to \boldsymbol{C} \times \boldsymbol{R} \times \boldsymbol{R}^3$, $\nabla_A = \nabla - iA$, $\Delta_A = {\nabla_A}^2$, $\rho = |u|^2$, $J = 2\operatorname{Im} \bar{u}\nabla_A u$. Physically, $u$ is the wave function of the particle, $(\phi, A)$ is the electro-magnetic potential, $\rho$ is the charge density, and $J$ is the current density. The system (MS) formally conserves at least two quantities, namely the total charge $\mathscr{Q} \equiv \|u\|_2^2$ and the total energy

$$\mathscr{E} \equiv \|\nabla_A u\|_2^2 + \frac{1}{2}\|\nabla\phi + \partial_t A\|_2^2 + \frac{1}{2}\|\operatorname{rot} A\|_2^2.$$

The system (MS) is gauge invariant and we study it in the Coulomb gauge $\operatorname{div} A = 0$, in which we can treat the system most easily. In this gauge, (1.2) and (1.3) become

$$-\Delta\phi = \rho, \quad \Box A + \nabla\partial_t\phi = J. \tag{1.4}$$

The first equation of (1.4) is solved as

$$\phi = \phi(u) = (-\Delta)^{-1}\rho = (4\pi|x|)^{-1} * |u|^2$$

and the term $\nabla\partial_t\phi$ in the second equation is dropped by operating the Helmholtz projection $P = 1 - \nabla \operatorname{div} \Delta^{-1}$ to the both sides of the equation. Therefore in the Coulomb gauge the system (MS) is rewritten as

$$i\partial_t u = (-\Delta_A + \phi(u))u, \tag{1.5}$$

$$\Box A = PJ, \tag{1.6}$$

which is referred to as (MS-C). To solve (MS-C) we should give the initial condition

$$(u(0), A(0), \partial_t A(0)) = (u_0, A_0, A_1) \tag{1.7}$$

in a direct sum of Sobolev spaces

$$X^{s,\sigma} = \{(u_0, A_0, A_1) \in H^s \oplus H^\sigma \oplus H^{\sigma-1}; \operatorname{div} A_0 = \operatorname{div} A_1 = 0\}.$$

Several authors study the Cauchy problem and the scattering theory for (MS-C). Nakamitsu-M. Tsutsumi [11] showed the time local well-posedness for (MS-C) in $X^{s,\sigma}$ with $s = \sigma = 3, 4, 5, \dots$. In fact, they treated the case of Lorentz gauge, but the Coulomb gauge case can be treated analogously. We remark that their condition can be refined as $s = \sigma > 5/2$ by the use of fractional order Sobolev spaces and the commutator estimate by Kato-Ponce [8]. Recently Nakamura-Wada [12] showed the time local well-posedness for wider class of $(s, \sigma)$ including the case $s = \sigma \geq 5/3$ (precisely see the remark for Theorem 1). On the other hand, Guo-Nakamitsu-Strauss [6] constructed a time global (weak) solution in $X^{1,1}$ although they did not show the uniqueness. Indeed, in the Coulomb gauge the energy takes the form

$$\mathscr{E} = \|\nabla_A u\|_2^2 + \frac{1}{2}\|\nabla\phi\|_2^2 + \frac{1}{2}\|\partial_t A\|_2^2 + \frac{1}{2}\|\nabla A\|_2^2,$$

and hence $\|(u, A, \partial_t A); X^{1,1}\|$ does not blow up. Therefore the global existence is proved by parabolic regularization and compactness method. For the scattering theory, the existence of modified wave operators was proved by Y. Tsutsumi [15], Shimomura [13], and Ginibre-Velo [4, 5]. However these results dose not mean the existence of global strong solution since their solution to (MS-C) exist only for $t \geq 0$ [13, 15] or for $t \geq T$ [4, 5], where $T$ is a sufficiently large positive number.

As we summarize above, there are many results for the Cauchy problem both at $t = 0$ or $t = \infty$. However there are no results concerning the global existence or blow up of strong solutions even for small data. The aim of this talk is to answer this problem. Shortly, we prove the global existence of unique strong solutions. To do this, we use a priori estimates derived from the conservation laws of charge and energy, and hence it is desirable to show the local well-posedness in lower regularity. The following theorem does not cover the result for the energy class $H^1$, but it is sufficient for our aim.

**Theorem 1.** *Let* $s \geq 11/8$, $\sigma > 1$ *and*

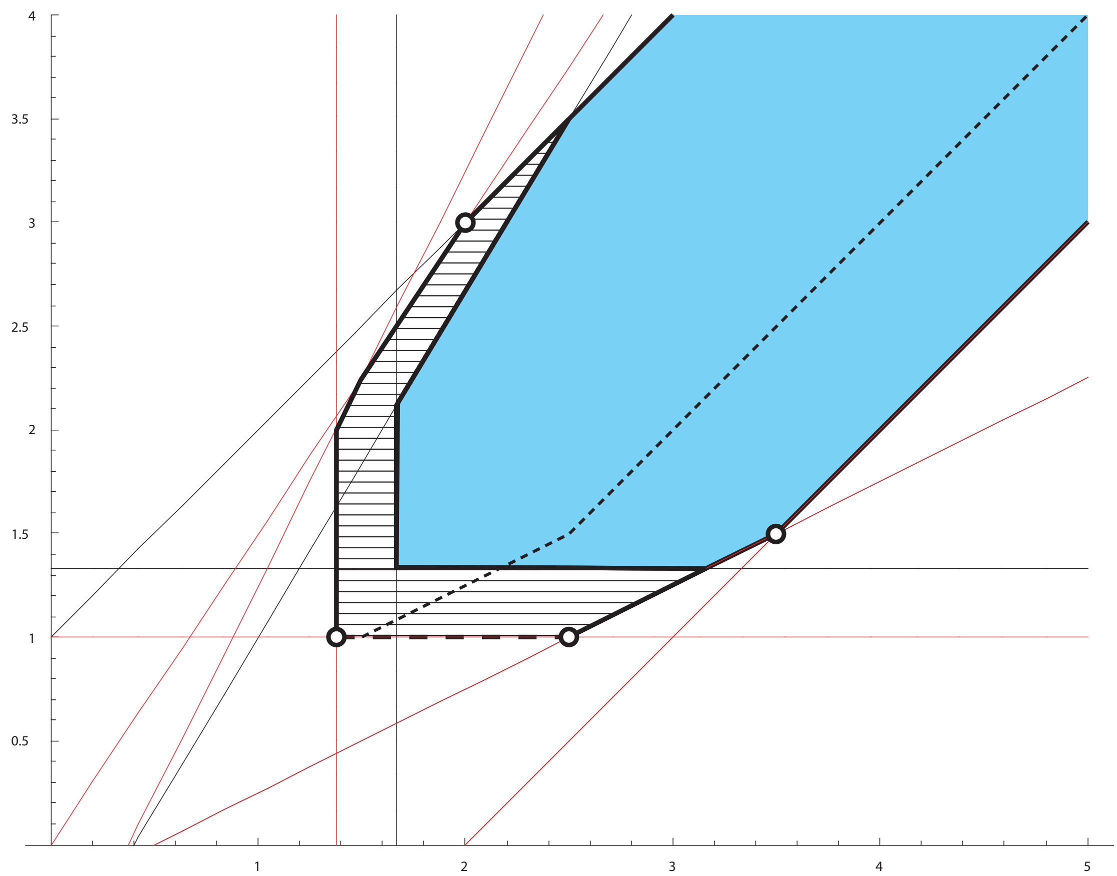
$$\max\{s-2; (2s-1)/4\} \leq \sigma \leq \min\{s+1; 3s/2; 2s-3/4\}$$

*with* $(s,\sigma) \neq (2,3),(7/2,3/2)$. *Then for any* $(u_0, A_0, A_1) \in X^{s,\sigma}$, *there exists* $T > 0$ *such that* (MS-C) *with initial condition* (1.7) *has a unique solution* $(u, A)$ *satisfying* $(u, A, \partial_t A) \in C([0,T]; X^{s,\sigma})$. *Moreover if* $s > 11/8$ *and* $\sigma \geq \max\{(s-1),(2s+1)/4\}$ *with* $(s,\sigma) \neq (5/2, 3/2)$, *then the map* $(u_0, A_0, A_1) \mapsto (u, A, \partial_t A)$ *is continuous as a map from* $X^{s,\sigma}$ *to* $C([0,T]; X^{s,\sigma})$.

*Remark.* (1) $T$ depends only on $s, \sigma$ and $\|(u_0, A_0, A_1); X^{s,\sigma}\|$.

(2) For any $s$ and $\sigma$ satisfying the assumption above for the unique existence of the solution, the map $(u_0, A_0, A_1) \mapsto (u, A, \partial_t A)$ is continuous in w*-sense. Namely if a sequence of initial data strongly converges in $X^{s,\sigma}$, then corresponding sequence of solutions also converges star-weakly in $L^\infty(0,T; X^{s,\sigma})$.

(3) In [12], we also assume $s \geq 5/3$ and $4/3 \leq \sigma \leq (5s-2)/3$ with $(s,\sigma) \neq (5/2, 7/2)$.




By relaxing the assumption for the local theory, we can show the global existence.

**Theorem 2.** *The solution obtained in Theorem* 1 *exists time globally.*

In this abstract we use the following notation: For a Banach space $X$ we put $L^q_T X = L^q(0,T;X)$. Similarly we use the abbreviation $C^m_T X = C^m([0,T];X)$. $M_T^{m,\sigma} = \bigcap_{j=0}^m C^j_T H^{\sigma-j}$ and its norm is defined as $\|A; M_T^{m,\sigma}\| = \max_{0\le j\le m} \|\partial_t^j A; L^\infty_T H^{\sigma-j}\|$.

## §2. Preliminaries

In this section we summarize lemmas used in the proof of Theorems 1 and 2. The first one is a covariant derivative estimate, whose proof is done by the use of the Gaglialdo-Nirenberg inequality.

**Lemma 1.** *Let $A \in \dot H^1 \cap L^6$ satisfy $\operatorname{div} A = 0$. Then the following estimates hold for any $v \in H^2$:*

$$\|\nabla_A v; H^1\| \lesssim \langle \|A; \dot H^1\|\rangle \|v; H^2\|, \tag{2.1}$$

$$\|v; H^2\| + \langle \|A; \dot H^1\|\rangle^4 \|v\|_2 \simeq \|\Delta_A v\|_2 + \langle \|A; \dot H^1\|\rangle^4 \|v\|_2. \tag{2.2}$$

Next we introduce Strichartz type estimates for Klein-Gordon equations (see for example [1–3, 14]).

**Lemma 2.** *Let $T > 0$, $\sigma \in \boldsymbol{R}$ and let $(q_j, r_j)$, $j = 0, 1$, satisfy $0 \le 2/q_j = 1 - 2/r_j < 1$. Then a solution $A$ to the equation $(\Box + 1)A = F$ satisfies the estimate*

$$\max_{k=0,1} \|\partial_t^k A; L^{q_0}_T H^{\sigma-k-2/q_0}_{r_0}\| \lesssim \|(A(0), \partial_t A(0)); H^\sigma \oplus H^{\sigma-1}\| + \|F; L^{q_1{}'}_T H^{\sigma-1+2/q_1}_{r_1{}'}\|. \tag{2.3}$$

Usual Strichartz estimates for Schrödinger equations does not match the equation (1.5) since we cannot avoid the loss of derivative coming from $2iA \cdot \nabla u$. In the present work we use a variation of Strichartz estimates introduced by Koch-Tzvetkov.

**Lemma 3.** *Let $T > 0$, $\alpha > 0$ and $s \in \boldsymbol{R}$. Then a solution $u$ to the equation*

$$i\partial_t u = -\Delta u + f, \quad 0 < t < T,$$

*satisfies the estimate*

$$\|u; L^2_T H^{s-\alpha}_6\| \lesssim \|u; L^\infty_T H^s\| + T^{1/2} \|f; L^2_T H^{s-2\alpha}\|. \tag{2.4}$$

This kind of estimates was first given by Koch-Tzvetkov [10] for the Benjamin-Ono equation, and it is Kenig-Koenig [9] who formulated the estimate as above. Kato [7] adapted this estimate for Schrödinger equations. However, in [7, 9], they need an extra assumption $u \in L^\infty_T H^{s+\epsilon}$ to prove (2.4), with the first term in the right-hand side replaced by $\|u; L^\infty_T H^{s+\epsilon}\|$.

**Lemma 4.** *Let $\sigma \ge 0$. Let $1 < p, p_1 < \infty$ and $1 < p_2 \le \infty$ satisfy $1/p = 1/p_1 + 1/p_2$. Then the following estimate holds valid:*

$$\|P(\bar u_1 \nabla u_2); H^\sigma_p\| \lesssim \|u_1; H^\sigma_{p_1}\| \|\nabla u_2\|_{p_2}. \tag{2.5}$$

## §3. Sketch of proof

In this section we shall sketch the proof of Theorem 2, from which we can also understand the essence of the proof of Theorem 1. For simplicity, we restrict our attention to the case $s = 2$, $0 < \sigma - 1 \ll 1$. In this section we fix a positive number $\delta$ so that $0 < \delta \le (\sigma - 1)/2$ and put $1/q = 1/2 - 2\delta/3$, $1/r = 2\delta/3$. We begin with the following a priori estimates.

**Lemma 5.** *Let* $(u, A, \partial_t A) \in C_T X^{s,\sigma}$ *be a solution to* (MS-C) *obtained in Theorem* 1. *Then the following estimates hold.*

$$\|(u, A, \partial_t A); L_T^\infty(H^1 \oplus \dot{H}^1 \oplus L^2)\| \le C, \tag{3.1}$$

$$\|A; L_T^\infty L^2\| \le C\langle T\rangle, \tag{3.2}$$

$$\|A; L_T^q L^r\| \le C\langle T\rangle^2, \tag{3.3}$$

$$\|u; L_T^2 H_6^{1/2-\delta}\| \le C\langle T\rangle^3, \tag{3.4}$$

$$\|A; M_T^{1,\sigma}\| \le C\langle T\rangle^5. \tag{3.5}$$

*The constants $C$ depend only on $\sigma$ and $\|(u_0, A_0, A_1); H^1 \oplus H^1 \oplus L^2\|$.*

*Proof.* We easily obtain (3.1)-(3.2) by the conservation laws of charge and energy. We obtain (3.3) by Lemma 2 together with (3.1)-(3.2). We obtain (3.4) by the use of Lemma 3, for

$$\begin{aligned}\|u; L_T^2 H_6^{1/2-\delta}\| &\lesssim \|u; L_T^\infty H^1\| + T^{1/2}\|2iA\cdot\nabla u + |A|^2 u + \phi u; L_T^2 H^{-2\delta}\| \\ &\lesssim \langle T\rangle \|u; L_T^\infty H^1\| \langle \|A; L_T^q L^r\| + \|A; L_T^\infty \dot{H}^1\|^2 + \|u; L_T^\infty H^1\|^2\rangle.\end{aligned}$$

Finally we obtain (3.5) by Lemma 2:

$$\|A; M_T^{1,\sigma}\| \lesssim \|(A_0, A_1); H^\sigma \oplus H^{\sigma-1}\| + \|A; L_T^1 H^{\sigma-1}\| + \|PJ; L_T^{6/5} H_{3/2}^{\sigma-2/3}\|$$

and the last term in the right-hand side is estimated by

$$\langle T\rangle \|u; L_T^\infty H^1\| \langle \|A; L_T^\infty H^1\|\rangle \|u; L_T^2 H_6^{1/2-\delta}\|.$$

Thus we have obtained the lemma. □

We proceed to the estimate of solutions to the following linear Schrödinger equation, namely in the following lemmas we regard $A$ and $u$ as known functions defined on $0 \le t \le T$:

$$i\partial_t v = (-\Delta_A + \phi(u))v, \quad 0 < t < T, \tag{3.6}$$

$$v(0) = v_0. \tag{3.7}$$

**Lemma 6.** *Let* $A \in M_T^{1,\sigma}$ *with* $\operatorname{div} A = 0$ *and let* $u \in C_T^\infty H^1$. *Let* $v \in C_T H^2$ *be a solution to* (3.6). *Then* $v \in L_T^2 H_6^{3/2-\delta}$ *and satisfies the estimate*

$$\|v; L_T^2 H_6^{3/2-\delta}\| \lesssim \langle T\rangle^m \langle \|A; L_T^\infty \dot{H}^1\|\rangle^m \langle \|A; L_T^q L^r\| \vee \|v; L_T^\infty H^1\|^2\rangle \|v; L_T^\infty H^2\|.$$

*Here $m$ is a positive number.*

*Proof.* Applying Lemma 3 to (3.6), we obtain

$$\|v; L_T^2 H_6^{3/2-\delta}\| \lesssim \|v; L_T^\infty H^2\| + T^{1/2}\|2iA\cdot\nabla v + |A|^2 v + \phi v; L_T^2 H^{1-2\delta}\|. \tag{3.8}$$

By the Leibniz rule we have $\|A\cdot\nabla v; H^{1-2\delta}\| \lesssim \|A; H_q^{1-2\delta}\|\|\nabla v\|_r + \|A\|_r\|\nabla v; H_q^{1-2\delta}\|$. Applying the estimate $\|\nabla v\|_r \lesssim \|v; H^2\|^\alpha \|v; H_6^{3/2-\delta}\|^{1-\alpha}$, $\alpha = 2\delta/(1-2\delta)$, derived from the Gagliardo-Nirenberg inequality, we obtain

$$\begin{aligned} T^{1/2}\|A\cdot\nabla v; L_T^2 H^{1-2\delta}\| &\lesssim \epsilon T^{1/2}\|v; L_T^2 H_6^{3/2-\delta}\| + \epsilon^{(\alpha-1)/\alpha} T\|A; L_T^\infty \dot H^1\|^{1/\alpha}\|v; L_T^\infty H^2\| \\ &\quad + T^{1-1/q}\|A; L_T^q L^r\|\|v; L_T^\infty H^2\|. \end{aligned}$$

We choose $\epsilon > 0$ so small that the first term in the right-hand side is absorbed in the left-hand side of (3.8). Another terms can be treated more easily. Thus we obtain the lemma. $\square$

**Lemma 7.** *Let $A \in M_T^{1,\sigma}$ with* $\operatorname{div} A = 0$ *and let $u \in C_T^\infty H^1$. Then there exists a unique solution to* (3.6)-(3.7) *belonging to $C_T H^2 \cap C_T^1 L^2$. Moreover the solution $v$ to* (3.6)-(3.7) *satisfies the following estimates*:

$$\begin{aligned} \|v; L_T^\infty H^2\| &\le C\|v_0; H^2\|\langle\|A; L_T^\infty \dot H^1\|\rangle^4 \\ &\quad \times \exp\{C\langle T\rangle^l \langle\|A; M_T^{1,\sigma}\| \vee \|A; L_T^q L^r\| \vee \|u; L_T^\infty H^1\|\rangle^m\}. \end{aligned} \tag{3.9}$$

*Here $l$ and $m$ are positive numbers.*

*Proof.* For simplicity we only prove the estimate (3.9). The conservation law $\|v(t)\|_2 = \|v_0\|_2$ immediately follows from the equation (3.6). Taking Lemma 1 into account, we estimate

$$\|v; H_A^2\| \equiv \|\Delta_A v\|_2 + \langle R\rangle^4\|v\|_2$$

instead of $\|v; H^2\|$, where $R \equiv \|A; L_T^\infty \dot H^1\|$. Taking the time derivative of $\Delta_A v$ and using the equation (3.6), we find the equation for $\Delta_A v$:

$$i\partial_t \Delta_A v = (-\Delta_A + \phi)\Delta_A v + 2\partial_t A\cdot\nabla_A v + [\Delta_A, \phi]v. \tag{3.10}$$

Therefore standard energy method shows that

$$\|v; L_T H_A^2\| \le \|v_0; H_{A_0}^2\| + \|2\partial_t A\cdot\nabla_A v + [\Delta_A, \phi]v; L_T^1 L^2\|.$$

Similarly as in the proof of Lemma 6, we have

$$\begin{aligned} \|\partial_t A\cdot\nabla_A v\|_2 &\lesssim \epsilon\|v; H_6^{3/2-\delta}\| \\ &\quad + \{\epsilon^{(\alpha-1)/\alpha}\|\partial_t A; H^{\sigma-1}\|^{1/\alpha} + \|\partial_t A; H^{\sigma-1}\|\|A\|_r\}\|v; H_A^2\|. \end{aligned} \tag{3.11}$$

We can easily handle the term $[\Delta_A, \phi]v$ by the Hardy-Littlewood-Sobolev inequality. Therefore

$$\begin{aligned}\|v; L_T^\infty H_A^2\| &\lesssim \|v_0; H_{A_0}^2\| \\ &+ \int_0^T \{\epsilon^{(\alpha-1)/\alpha}\|\partial_t A; H^{\sigma-1}\|^{1/\alpha} + \|\partial_t A; H^{\sigma-1}\|\|A\|_r + \|u; H^1\|^2\}\|v; H_A^2\|dt \\ &+ T^{1/2}\epsilon\|v; L_T^2 H_6^{3/2-\delta}\|.\end{aligned}$$

Taking Lemma 6 into account, we choose the positive number $\epsilon$ so small that the last term in the right-hand side is absorbed in the left-hand side. Then we obtain an integral inequality for $\|v; H_A^2\|$. Applying the Gronwall inequality we obtain (3.9). □

*Proof of Theorem* 2. The solution $(u, A)$ to (MS-C) clearly satisfies the estimate (3.9) with $v = u, v_0 = u_0$. Therefore the global existence follows from Lemma 5. □

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