PARTIAL REGULARITY OF WEAK SOLUTIONS TO DEGENERATE PARABOLIC SYSTEMS OF POROUS MEDIUM TYPE

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1. Introduction

We consider the following reaction-diffusion equation:

$$(\mathrm{KS})_m \begin{cases} u_t & = \Delta u^m - \nabla \cdot \big(u^{q-1}\nabla v\big), \quad x \in \mathbb{R}^N,\ t > 0, \\ 0 & = \Delta v - \gamma v + u, \quad x \in \mathbb{R}^N,\ t > 0, \\ u(x,0) & = u_0(x), \quad x \in \mathbb{R}^N. \end{cases}$$

This equation is often called the Keller-Segel model describing the motion of the chemotaxis molds. Here $u(x,t)$ and $v(x,t)$ denote the density of amoebae and the concentration of the chemo-attractant, respectively. We refer to Keller-Segel [4], Horstman[3], Suzuki[10].

Throughout this talk, we impose the following assumption:

Assumption The space dimension $N \geq 3$ and the coefficient $\gamma > 0$. Moreover, $m > 1,\ q \geq 2$ satisfy

$$q = m + \frac{2}{N}.$$

The initial data u_0 is a non-negative function satisfying

$$u_0 \in L^1 \cap L^\infty(\mathbb{R}^N) \quad \text{with } u_0^m \in H^1(\mathbb{R}^N).$$

Our definition of a weak solution now reads:

Definition 1 *Let the Assumption hold. A pair (u,v) of non-negative functions defined in $\mathbb{R}^N \times [0,T)$ is called a weak solution of $(\mathrm{KS})_m$ on $[0,T)$ if*

(i) $u \in L^\infty(0,T;L^1(\mathbb{R}^N)) \cap L^\infty(0,T';L^\infty(\mathbb{R}^N))$ *for all* T' *with* $0 < T' < T$;

(ii) $\nabla u^m \in L^2(0,T;L^2(\mathbb{R}^N))$;

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(iii) $v \in L^\infty(0,T;H^1(\mathbb{R}^N))$;

(iv) (u,v) *satisfies the following identities:*

$$\begin{aligned}
\int_0^T \int_{\mathbf{R}^N} \left(\nabla u^m \cdot \nabla \varphi - u^{q-1}\nabla v \cdot \nabla \varphi - u\varphi_t\right)\, dxdt &= \int_{\mathbf{R}^N} u_0(x)\varphi(x,0)\, dx,\\
\int_{\mathbf{R}^N} (\nabla v \cdot \nabla \psi + v \cdot \psi - u \cdot \psi)\, dx &= 0 \qquad a.a.\ t \in [0,T)
\end{aligned}$$

for all $\varphi \in W^{1,2}(0,T;L^2(\mathbb{R}^N)) \cap L^2(0,T;W^{1,2}(\mathbb{R}^N))$ *satisfying* $\varphi(\cdot,t) = 0$ *for all* $t \in [T',T]$ *with some* $0 < T' < T$, *and all* $\psi \in H^1(\mathbb{R}^N)$.

Concerning the time local existence of weak solutions to $(\mathrm{KS})_m$, the following result can be shown by a slight modification of the argument developed by the author [12, Theorem 1.1].

Proposition 1.1. **(local existence of weak solution and its uniform L^∞-bound)** *Let the Assumption hold. Then there exist* T_0 *and a weak solution* (u,v) *of* $(KS)_m$ *on* $[0,T_0)$ *with the following additional properties:*

$$\|u(t)\|_{L^1(\mathbf{R}^N)} = \|u_0\|_{L^1(\mathbf{R}^N)} \qquad \text{for all } 0 \le t < T_0; \tag{1.1}$$

$$(u^{\frac{m+1}{2}})_t \in L^2(0,T_0;L^2_{loc}(\mathbb{R}^N)). \tag{1.2}$$

Such an interval T_0 *of local existence can be taken as* $T_0 = \left(\|u_0\|_{L^\infty(\mathbf{R}^N)} + 2\right)^{-q}$, *and the weak solution* $u(t)$ *above satisfies the following estimate:*

$$\|u(t)\|_{L^\infty(\mathbf{R}^N)} \le \|u_0\|_{L^\infty(\mathbf{R}^N)} + 2 \qquad \text{for all } t \in [0,T_0). \tag{1.3}$$

Next, we state the main theorem on the ε-regularity for the weak solutions of $(\mathrm{KS})_m$.

Theorem 1.2. **(ε-regularity)** *Let the Assumption hold. Then there exists a positive number* ε_0 *depending only on* N *and* m *with the following property:*

Suppose that (u,v) *is an arbitrary weak solution of* $(KS)_m$ *on* $[0,T)$ *in Definition 1 with the additional properties (1.1)–(1.2). If it holds*

$$\limsup_{\rho \to 0} \left(\sup_{0<t<T} \int_{B(x_0,\rho)} u(x,t)\, dx \right) \le \varepsilon_0 \tag{1.4}$$

for some $x_0 \in \mathbb{R}^N$, *then there exists* $\rho_0 > 0$ *such that*

$$\sup_{(x,t)\in B(x_0,\rho_0)\times(0,T)} u(x,t) < \infty.$$

Remark 1. (1) It should be noted that the quantity $\sup_{0<t<\infty} \|u(t)\|_{L^{\frac{N(q-m)}{2}}(\mathbf{R}^N)}$ is invariant under the change of scaling associated to $(\mathrm{KS})_m$ with $\gamma = 0$. In fact, if (u,v)

solves $(\mathrm{KS})_m$ with $\gamma = 0$, then (u_λ, v_λ) is also a solution for all $\lambda > 0$, where

$$\begin{cases} u_\lambda(x,t) := \lambda^2 u\Big(\lambda^{q-m}x, \lambda^{2(q-1)}t\Big), \\ v_\lambda(x,t) := \lambda^{2(m-q+1)} v\Big(\lambda^{q-m}x, \lambda^{2(q-1)}t\Big). \end{cases}$$

The scaling invariance in $L^{\frac{N(q-m)}{2}}(\mathbb{R}^N)$ means that

$$\sup_{0<t<\infty} \|u_\lambda(t)\|_{L^{\frac{N(q-m)}{2}}(\mathbb{R}^N)} = \sup_{0<t<\infty} \|u(t)\|_{L^{\frac{N(q-m)}{2}}(\mathbb{R}^N)} \qquad \text{for all } \lambda > 0. \tag{1.5}$$

In particular, for $q = m + \frac{2}{N}$, the above (1.5) is equivalent to

$$\sup_{0<t<\infty} \|u_\lambda(t)\|_{L^1(\mathbb{R}^N)} = \sup_{0<t<\infty} \|u(t)\|_{L^1(\mathbb{R}^N)} \qquad \text{for all } \lambda > 0$$

since $\frac{N(q-m)}{2} = 1$. Therefore, we may say that (1.4) is a natural condition concerning the partial regularity of weak solutions to $(\mathrm{KS})_m$.

(2) The critical exponent such as dividing the situation into global existence and blow-up of solutions was originally found in the following Fujita type equations:

$$\frac{\partial u}{\partial t} = \Delta u^m + u^q \tag{F}$$

with $m \geq 1,\ q > 1$. It is well-known that the exponent $q = m + \frac{2}{N}$ is the critical one. So, our results [11]–[15] may be regarded as an extension of the critical exponent from (F) to $(\mathrm{KS})_m$.

As an application of the ε-regularity theorem as Theorem 1.2, we characterized the asymptotic behavior of blow-up solutions to $(\mathrm{KS})_m$. For that purpose, let us introduce definitions for the *blow-up time* and the *blow-up point*.

Definition 2 *Let (u, v) be the weak solution of $(KS)_m$ on $[0, T)$ in Definition 1.*

(i) (blow-up time) We say that u blows up at the time $T < \infty$ if

$$\limsup_{t \to T-0} \|u(t)\|_{L^\infty(\mathbb{R}^N)} = \infty. \tag{1.6}$$

Such a T is called a blow-up time of u.

(ii) (blow-up point) Let T be a blow-up time of u. We call $x_0 \in \mathbb{R}^N$ a blow-up point of u at the time T if there exists $\{(x_n, t_n)\}_{n=1}^\infty \subset \mathbb{R}^N \times (0, T)$ such that

$$x_n \to x_0, \quad t_n \to T, \quad \text{and} \quad u(x_n, t_n) \to \infty \quad \text{as } n \to \infty.$$

Remark 2. Since the time interval T_0 of the local weak solution can be expressed by $\|u_0\|_{L^\infty(\mathbb{R}^N)}$ as in Proposition 1.1, we see that the weak solution u of $(\mathrm{KS})_m$ on $[0, T)$ can be continued beyond $t = T$ provided

$$\limsup_{t \to T-0} \|u(t)\|_{L^\infty(\mathbb{R}^N)} < \infty.$$

Hence, the maximal existence time $T_{\max}$ of the weak solution u of $(\mathrm{KS})_m$ is, in particular, a blow-up time of the weak solution u of $(\mathrm{KS})_m$.

An immediate consequence of Theorem 1.2 is the following characterization of both the blow-up points x_0 and the time T.

Corollary 1.3. *Let the Assumption hold. Suppose that (u, v) is the weak solution of $(KS)_m$ on $[0, T)$ with the additional properties (1.1)–(1.2). Let T be the blow-up time of the weak solution u of $(KS)_m$. Then, for any blow-up point $x_0 \in \mathbb{R}^N$ of u of $(KS)_m$ at the time T, it holds that*

$$\sup_{0<t<T} \int_{B(x_0,\rho)} u(x,t)\, dx \;>\; \varepsilon_0 \qquad \text{for all } \rho > 0, \tag{1.7}$$

where ε_0 is the same constant given by Theorem 1.2.

There are several methods to construct weak solutions (u, v) of $(\mathrm{KS})_m$ on some interval $[0, T)$. If we adopt a special construction of the weak solution, then the corresponding ε-regularity theorem is established with a sharper constant than ε_0.

Theorem 1.4. *Let the Assumption hold. Suppose that (u, v) is the weak solution of $(KS)_m$ on $[0, T)$ given by Proposition 1.1. For such u, and a positive constant κ, we define the set $\Omega_\kappa(t)$ by*

$$\Omega_\kappa(t) := \{x \in \mathbb{R}^N; u(x,t) > \kappa\} \qquad \text{for } 0 \le t < T. \tag{1.8}$$

Assume that there are positive constants κ and $0 < \alpha < T$ such that the set $\Omega^*_\kappa := \bigcap_{T-\alpha<t<T} \Omega_\kappa(t)$ is a domain in $\mathbb{R}^N$.
If u satisfies

$$\limsup_{\rho\to+0} \left(\sup_{T-\alpha<t<T} \int_{B(x_0,\rho)} u(x,t)dx \right) < \left(\frac{m\pi N^3}{N-1} \right)^{\frac{N}{2}} \frac{\Gamma(N/2)}{\Gamma(N)} \quad (\;=: \; \alpha_{N,m} \;) \tag{1.9}$$

for some $x_0 \in \Omega^*_\kappa$, then there exists $\rho_0 > 0$ such that

$$\sup_{(x,t)\in B(x_0,\rho_0)\times(T-\alpha,T)} u(x,t) < \infty. \tag{1.10}$$

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