Blow-up solutions of modified Schrödinger maps

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We consider the system of the nonlinear Schrödinger equations in two space dimensions

$$\text{(MS)}\begin{cases} iD_0u_1 = -\sum_{k=1}^2 D_k^2 u_1 - 4i\sum_{k=1}^2 \mathrm{Im}(u_1\bar{u}_k)u_k, \\ iD_0u_2 = -\sum_{k=1}^2 D_k^2 u_2 - 4i\sum_{k=1}^2 \mathrm{Im}(u_2\bar{u}_k)u_k, \end{cases}$$

where u_1, u_2 are complex valued functions, $D_\mu = \partial_\mu + iA_\mu$ is covariant derivative associated with vector potentials A_μ which are defined by

$$F_{12} = 4\mathrm{Im}(u_1\bar{u}_2), \qquad \nabla\cdot A = 0,$$
$$-\Delta A_0 = 4\sum_{j,k=1}^{2} \partial_j\partial_k \mathrm{Re}(u_j\bar{u}_k) - 2\Delta(|u_1|^2 + |u_2|^2),$$

where $F_{12} = \partial_1A_2 - \partial_2A_1$ and $\nabla\cdot A = \partial_1A_1 + \partial_2A_2$. The system above is called the modified Schrödinger map which was derived by Nahmod-Stefanov-Uhlenbeck [10] from Schrödinger maps from $\mathbb{R}^{2+1}$ to the unit sphere $\mathbb{S}^2$ by using appropriate gauge change. In fact, Schrödinger map $\phi : \mathbb{R}\times\mathbb{R}^2 \to (\mathbb{C},\, g\,dzd\bar{z})$ (where $g(z,\bar{z}) = (1+|z|^2)^{-2}$) is given by

$$\frac{\partial\phi}{\partial t} = i\sum_{j=1}^{2}\left(\partial_j - 2\frac{\bar{\phi}\partial_j\phi}{1+|\phi|^2}\right)\partial_j\phi, \tag{0.1}$$

where $\bar{\phi}$ is the complex conjugate of ϕ. Its formal equivalence to Landau-Lifshitz equation [2] can be seen by applying the stereographic projection from $\mathbb{C}_\infty$, the extended complex plane to $\mathbb{S}^2$,

$$z\in\mathbb{C} \to \left(\frac{2\mathrm{Re}z}{1+|z|^2}, \frac{2\mathrm{Im}z}{1+|z|^2}, \frac{1-|z|^2}{1+|z|^2}\right) = (s_1, s_2, s_3) \in \mathbb{S}^2,$$
$$\partial_t s = s\times\Delta s, \qquad |s| = 1.$$

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They applied the gauge transformation

$$u_j = e^{i\psi} \frac{\partial_j \phi}{1 + |\phi|^2}, \qquad j = 1, 2 \tag{0.2}$$

to derive (MS) by choosing the gauge ψ appropriately.

The modified Schrödinger map is invariant with respect to the following scaling

$$u(t, x) \mapsto \lambda u(\lambda^2 t, \lambda x), \quad \lambda > 0.$$

Then it is expected that the critical space for the local well-posedness of the Cauchy problem is $L^2(\mathbb{R}^2)$. We also notice that (MS) conserves the L^2 norm. The Cauchy problem for (MS) has been studied by several authors [6, 7, 9, 10].

Here we are interested in the blow-up solution of (MS) which is the nonlinear Schrödinger equations coupled with vector potentials A_μ. The blow-up problem for nonlinear Schrödinger equations has been studied extensively [1, 3, 8, 11, 12]. In particular, it was proved in [1] that there is a blow-up solution to Chern-Simons-Schrödinger equations which resemble (MS). The Chern-Simons-Schrödinger equations [4, 5] have variational structure which was used to show a blow-up solution by using functional approach. However our system (MS) does not have variational structure as far as we know. Here we find the modified pseudo-conformal invariance of (MS) which is crucial to construct explicit blow-up solutions.

Remark. A priori estimate and the estimate on the time of existence on the smooth solution to (MS) are made use of in order to construct the low regularity solution to the Schrödinger map. However, as is pointed in [10], it is not possible to use directly the information of the (MS) to the original Schrödinger map.

References

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