障害物のある振動方程式の解のミニマイジング・ムーブメント法による構成

菊地光嗣（静岡大工）

1 Introduction

In [5] M. Schatzman treats a problem which describs movement of a string that hits to an obstacle. This problem is formulated as in the following way. In [5] a slightly general obstacle is considered. However for the sake of simplicity we consider the case that the obstacle is a plane, just like as a table.

Given $u_0 \in W^{1,2}(0,1)$ and $v_0 \in L^2(0,1)$ with $u_0 \geq 0$ and $u_0(0) = u_0(1) = 1$, we treat a second order hyperbolic differential inequality

$$u_{tt} - u_{xx} \geq 0 \tag{1}$$

in the sense of distributions and

$$\text{spt } (u_{tt} - u_{xx}) \subset \{u = 0\} \tag{2}$$

with initial conditions

$$u(0,x) = u_0, \qquad \frac{\partial u}{\partial t}(0,x) = v_0 \tag{3}$$

and a boundary condition

$$u(t,0) = u(t,1) = 1. \tag{4}$$

A weak solution to (1)–(4) is defined as follows:

Definition 1 A function $u : (0,T) \to L^2(0,1)$ is said to be a weak solution to (1)–(4) in $(0,T)$ if

i) $u \in W^{1,2}((0,T) \times (0,1))$, $u(t,x) \geq 0$ for $\mathcal{L}^2$-a.e. (t,x)

ii) $\text{s-}\lim_{t \searrow 0} u(t) = u_0$ in $L^2(0,1)$

iii) $u(t,0) = u(t,1) = 1$

iv) for any $\phi \in C_0^0([0,T); L^2(0,1)) \cap W_0^{1,2}((0,T) \times (0,1))$ with $\phi \geq 0$,

$$-\int_0^T \int_0^1 u_t(t)\phi_t(t)dxdt + \int_0^T \int_0^1 u_x \phi_x dxdt - \int_0^1 v_0 \phi dx \geq 0.$$

v) for any $\phi \in C_0^0([0,T); L^2(0,1)) \cap W_0^{1,2}((0,T) \times (0,1))$ with spt $\phi \subset (\{u = 0\})^c$,

$$-\int_0^T \int_0^1 u_t(t)\phi_t(t)dxdt + \int_0^T \int_0^1 u_x \phi_x dxdt - \int_0^1 v_0 \phi dx = 0.$$

In [5] M. Schatzman solves this equation in a slightly classical way. Moreover uniqueness is also proved under an assumption that a solution satisfies an equality which assures the energy conservation law. In [3] K. Maruo constructs a solution to this problem by the use of Yosida approximation. The purpose of this talk is to construct a solution to this problem in minimizing movement method. Readers should remark that in general approximation by minimizing movement method is different from Yosida approximation (compare to [1]).

2 Minimizing movement method

We define a functional $\Phi : L^2(0,1) \to [0,\infty]$ by

$$\Phi(u) = \begin{cases} 0 & \text{if} \quad u(x) \geq 0 \text{ for each } x \\ \infty & \text{if} \quad \text{otherwise.} \end{cases}$$

Put $J(u) = \frac{1}{2}\int_0^1 |\nabla u|^2 dx + \Phi(u)$ if $u \in W^{1,2}(0,1) \cap D(\Phi)$ with $u(0) = u(1) = 1$, $= \infty$ if otherwise. Note that $u_0 \in D(J)$. For a positive number h we construct a sequence $\{u_l\}_{l=-1}^{\infty}$ in the following way. For $l = 0$ we let u_0 be as above and for $l = -1$ we set $u_{-1} = u_0 - hv_0$. For $l \geq 1$, u_l is defined as the minimizer of the functional

$$\mathcal{F}_l(u) = \frac{1}{2h^2}\|u - 2u_{l-1} + u_{l-2}\|^2 + J(u)$$

in $D(J)$, namely, in $W^{1,2}(0,1) \cap D(\Phi)$ with $u(0) = u(1) = 1$. The existence of the minimizer is assured by lower semicontinuity of J and its boundedness from below. By the use of convexity of J we have

Lemma 1 (Energy inequality)

$$\frac{1}{2h^2}\|u_l - u_{l-1}\|^2 + J(u_l) \leq \frac{1}{2}\|v_0\|^2 + J(u_0).$$

Next we define approximate solutions $u^h(t)$ and $\overline{u}^h(t)$ for $t \in (-h,\infty)$ as follows: for $(l-1)h < t \leq lh$

(5) $$u^h(t,x) = \frac{t-(l-1)h}{h}u_l + \frac{lh-t}{h}u_{l-1}$$

and

(6) $$\overline{u}^h(t) = u_l.$$

Then Lemma 1 shows

(7) $$\frac{1}{2}\int_0^1 |u_t^h(t)|^2 dx + J(\overline{u}^h(t)) \leq \frac{1}{2}\int_0^1 |v_0|^2 dx + J(u_0)$$

for each $t \in \bigcup_{l=0}^{\infty}((l-1)h, lh)$.

Proposition 1 *It holds that*

1). $\{\|u_t^h\|_{L^\infty((0,\infty);L^2(0,1))}\}$ *is uniformly bounded with respect to* h

2). $\{\|(\overline{u}^h)_x\|_{L^\infty((-h,\infty);L^2(0,1))}\}$ *is uniformly bounded with respect to* h

3). $\overline{u}^h(t,x) \geq 0$ *for each* x *and* $\mathcal{L}^1$*-a.e.* t

4). $\{\|(u^h)_x\|_{L^\infty((0,\infty);L^2(0,1))}\}$ *is uniformly bounded with respect to* h

5). $u^h(t,x) \geq 0$ *for each* x *and* $\mathcal{L}^1$*-a.e.* t

Then there exist a sequence $\{h_j\}$ *with* $h_j \to 0$ *as* $j \to \infty$ *and a function* u *such that*

4). *for any* $T > 0$, u^h *converges to* u *as* $j \to \infty$ *weakly star in* $L^\infty((0,T); L^2(0,1))$

5). u_t^h *converges to* u_t *as* $j \to \infty$ *weakly star in* $L^\infty((0,\infty); L^2(0,1))$

6). $(u^h)_x$ *converges to* u_x *as* $j \to \infty$ *weakly star in* $L^\infty((0,\infty); L^2(0,1))$

7). *for any* $T > 0$, u^h *converges to* u *as* $j \to \infty$ *strongly in* $L^\infty((0,T); L^2(0,1))$

8). *for any* $T > 0$, $\overline{u}^h$ *converges to* u *as* $j \to \infty$ *strongly in* $L^\infty((0,T); L^2(0,1))$

9). $s\text{-}\lim_{t \searrow t_0} u(t) = u_0$ *in* $L^2(0,1)$.

3 Main Theorem

Our main theorem is as follows:

Theorem 1 *The function* u *as in Proposition* 1 *is a weak solution to* (1)–(4).

Outline of Proof. Since u_l is the minimizer of $\mathcal{F}_l(v)$, we have $\partial\mathcal{F}_l(u_l) = \dfrac{u_l - 2u_{l-1} + u_{l-2}}{h^2} + \partial J(u_l) \ni 0$. Since $J(u) = \frac{1}{2}\int_0^1 |\nabla u|^2 dx + \Phi(u)$, we have $\dfrac{u_l - 2u_{l-1} + u_{l-2}}{h^2} - \triangle u_l + \partial\Phi(u_l) \ni 0$. Namely, noting (5) and (6), we have, for each h,

$$\Phi(\overline{u}^h(t) + \phi) - \Phi(\overline{u}^h(t)) \geq -\int_0^1 \frac{u_t^h(t) - u_t^h(t-h)}{h}\phi(x)dx - \int_0^1 \nabla\overline{u}^h \nabla\phi dx \tag{8}$$

for $\mathcal{L}^1$-a.e. $t \in (0,\infty)$. Proposition 1 implies u_t^h and $(\overline{u}^h)_x$ converge weakly star to u_t and u_x, respectively, in $L^\infty((0,\infty); L^2(0,1))$. Thus, if $\phi \geq 0$, since $\Phi(\overline{u}^h + \phi) = \Phi(\overline{u}^h) = 0$, (8) implies iv) of the definition of a solution by letting $h \to 0$.

Lemma 2 *Let* $\varphi \in L^2(0,1)$ *and suppose that* $\varphi' \in L^2(0,1)$ *and* $\varphi(0) = 0$. *Then* $\|\varphi\|_{L^\infty(0,1)} \leq \sqrt{2}\|\varphi\|_{L^2(0,1)}^{1/2}\|\varphi'\|_{L^2(0,1)}^{1/2}$.

Since we have

$$u^h(t) - u^h(t') = \int_{t'}^t u_t^h(s)ds, \tag{9}$$

for each t, $t' \geq 0$, Proposition 1 1) implies

$$\|u^h(t) - u^h(t')\|_{L^2(0,1)} \leq C|t - t'|, \tag{10}$$

where C is independent of h. In the sequel C always denotes a generic constant which is independent of C. By proposition 1 4) we have

$$\|(u^h)_x(t) - (u^h)_x(t')\|_{L^2(0,1)} \leq C \tag{11}$$

By (10), (11), and Lemma 2 we have

$$\|u^h(t) - u^h(t')\|_{L^\infty(0,1)} \leq C|t - t'|^{1/2}. \tag{12}$$

This fact implies

$$\begin{aligned}|u^h(t,x)-u^h(s,y)| &\le |u^h(t,x)-u^h(t,y)|+|u^h(t,y)-u^h(s,y)|\\ &= |\int_y^x u_x^h(t,\xi)d\xi|+|u^h(t,y)-u^h(s,y)|\\ &\le \|u_x^h\|_{L^\infty((0,T);L^2(0,1))}|x-y|^{1/2}+C|t-s|^{1/2}\end{aligned}$$

Thus by Proposition 1 4) u^h is equicontinuous in $(0,T)\times(0,1)$ with respect to h. Furthermore, letting $s=0$ and $y=0$, we find $\{u^h\}$ is uniformly bounded in $L^\infty((0,T)\times(0,1))$. Hereby we have by Ascoli-Arzela theorem that, passing to a further subsequence if necessary, $\{u^h\}$ converges uniformly in $(0,T)\times(0,1)$ to u. Let $\phi \in C_0^0([0,T);L^2(0,1)) \cap W_0^{1,2}((0,T)\times(0,1))$ satisfy spt $\phi \subset (\{u=0\})^c = \{u>0\}$. Here remark that u is continuous with respect to t and x. Thus there should be a positive constant σ such that $u \ge \sigma$ in spt ϕ. We may suppose that $\sup|\phi| \le \sigma$. Since $u^h(t,x)$ converges uniformly to $u(t,x)$, $|u(t,x)-u^h(t,x)| < \frac{1}{2}\sigma$ if h is sufficiently small. Thus we have

$$u^h+\phi = u+\phi+u^h-u \ge u-|\phi|-|u-u^h| \ge \sigma-\frac{1}{2}\sigma-\frac{1}{2}\sigma = 0.$$

Hence $u^h+\phi \ge 0$ in $(0,T)\times(0,1)$. Noting that $\overline{u}^h(t,x)=u^h(lh,x)$ for $(l-1)h<t\le lh$, we find $\overline{u}^h+\phi \ge 0$ in $(0,T)\times(0,1)$. Hence (8) implies

$$0=\Phi(\overline{u}^h+\phi)-\Phi(\overline{u}^h) \ge -\int_0^1 \frac{u_t^h(t)-u_t^h(t-h)}{h}\phi(t,x)dx-\int_0^1(\overline{u}^h)_x\phi_x dx$$

for $\mathcal{L}^1$-a.e. t. Replacing ϕ with $-\phi$ we have the converse inequality and thus, for $\mathcal{L}^1$-a.e. t,

$$-\int_0^1 \frac{u_t^h(t)-u_t^h(t-h)}{h}\phi(t,x)dx-\int_0^1(\overline{u}^h)_x\phi_x dx = 0.$$

Integrating over $(0,T)$ and letting $h\to 0$, we have v) of the definition of a solution.

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