Global Solution to the Initial-Boundary Value Problem for a Spherically Symmetric Stellar Model with Rigid Core

梅原 守道 (Morimichi UMEHARA), 谷 温之 (Atusi TANI)
慶應義塾大学 理工学部
Department of Mathematics, Keio University

1 Introduction.

We consider the three-dimensional spherically symmetric motion of a compressible, viscous and heat-conductive gas in the free-boundary case. This classical model has been studied in many astrophysical literatures (see for example, [2]). Many stars being in way of their evolution have the core in the centre composed by the heavy chemical elements (the helium, et al.) produced by the burning of the light gas (the hydrogen) in the outer part of that. Under this situation it is natural to take into account the self-gravitation of gas and the potential force of the core, which drive the motion of gas, and also, at the high temperature stage, radiation phenomenon and the energy producing process inside the medium, that is, the gas consists of a reacting mixture and the combustion process is current.

For technical reason, let the core be a rigid sphere. The motion of our model is described by the following four equations in the polar coordinate system of Lagrangian form corresponding to the conservation laws of mass, momentum and energy, and an equation of reaction-diffusion type:

$$\begin{cases} v_t = (r^2u)_x \\ u_t = r^2\left(-p+\zeta\dfrac{(r^2u)_x}{v}\right)_x - G\,\dfrac{x+M_0}{r^2}, \\ e_t = \left(\dfrac{r^4\kappa\theta_x}{v}\right)_x + \left(-p+\zeta\dfrac{(r^2u)_x}{v}\right)(r^2u)_x - 4\mu(ru^2)_x + \lambda\phi z, \\ z_t = \left(\dfrac{dr^4z_x}{v^2}\right)_x - \phi z \end{cases} \tag{1.1}$$

in $(0,1)\times(0,\infty)\ \big(\ni(x,t)\big)$. Here the specific volume $v=v(x,t)$, the velocity $u=u(x,t)$, the absolute temperature $\theta=\theta(x,t)$ and the mass fraction of the reactant $z=z(x,t)$ are the unknown functions, and positive constants μ, ζ, d and λ are the coefficients of viscosity, bulk viscosity, which satisfy the relation $3\zeta-4\mu>0$, the species diffusion and the difference in heat between the reactant and the product. M_0 and G are mass of the core and the Newtonian gravitational constant, respectively. $r=r(x,t)$ is the

Eulerian position of the gas given by the formula

$$r = r(x,t) = \left({R_0}^3 + 3\int_0^x v(\xi,t)\,\mathrm{d}\xi \right)^{1/3}, \quad r_t = u, \quad r_x = \frac{v}{r^2},$$

where R_0 is the radius of the core. The rate function $\phi = \phi(\theta)$ is defined by the Arrhenius law

$$\phi(\theta) = K\theta^\beta \mathrm{e}^{-A/\theta},$$

where positive constants A and K are the activation energy and coefficient of rate of reactant, respectively. β is a non-negative number. Pressure $p = p(v,\theta)$ and internal energy $e = e(v,\theta)$ are given by $p = p_G + p_R$ and $e = e_G + e_R$, respectively, where $p_G = p_G(v,\theta)$ and $e_G = e_G(v,\theta)$ are the *gaseous* (elastic) ones. For technical reason, we assume the gas is ideal:

$$p_G = R\frac{\theta}{v}, \qquad e_G = c_{\mathrm{v}}\theta$$

with the perfect gas constant R and the specific heat at constant volume c_{v} (positive constants). Since it is widely accepted to assume the gas, constituents of the star, to be "black body", the influence of radiation is expressed by the *radiative* terms $p_R = p_R(\theta)$ and $e_R = e_R(v,\theta)$, which are given by the Stefan-Boltzmann law (see for example, [2])

$$p_R = \frac{a}{3}\theta^4, \qquad e_R = av\theta^4$$

with the Stefan-Boltzmann constant $a > 0$. We also assume the conductivity $\kappa = \kappa(v,\theta)$ has the following form:

$$\kappa = \kappa_1 + \kappa_2 v\theta^q,$$

where κ_1, κ_2 and q are positive constants.

We impose the boundary conditions for $t > 0$

$$\begin{cases} \left(-p + \zeta\dfrac{(r^2u)_x}{v} - 4\mu\dfrac{u}{r} \right)\Bigg|_{x=1} = -p_e, \\ u|_{x=0} = 0, \\ (\theta_x, z_x)|_{x=0,1} = (0,0), \end{cases} \tag{1.2}$$

the initial condition for $x \in [0,1]$

$$(v,u,\theta,z)|_{t=0} = \big(v_0(x), u_0(x), \theta_0(x), z_0(x)\big). \tag{1.3}$$

Here we assume the compatibility conditions

$$\begin{cases} \left(-p_0 + \zeta\dfrac{({r_0}^2u_0)'}{v_0} - 4\mu\dfrac{u_0}{r_0} \right)\Bigg|_{x=1} = -p_e, \\ u_0(0) = {\theta_0}'(0) = {\theta_0}'(1) = {z_0}'(0) = {z_0}'(1) = 0 \end{cases} \tag{1.4}$$

with $p_0 := R\theta_0/v_0 + (a/3){\theta_0}^4$ and $r_0 := ({R_0}^3 + 3\int_0^x v_0(\xi)\,\mathrm{d}\xi)^{1/3}$.

The global unique existence problem for compressible viscous polytropic ideal gas with large initial data is firstly solved by Kazhikhov-Shelukhin [8] in one-dimension under Dirichlet boundary condition with respect to the velocity. After that, in one-dimensional case, many works has been done including Nagasawa [11] studying the global existence and the asymptotic behavior in the free-boundary case for the polytropic ideal gas. Bebernes-Bressan [1], Chen [3] considered models for a reacting mixture. Many authors (see for exmple, [7]) studied models having some general pressure, internal energy and conductivity applicable to the situation with low order θ-dependency of p, e and κ. Ducomet [5] and Umehara-Tani [13] are extended these results to the one with higher order θ-dependency of p and e, which usually seems to correspond to some large q in κ for getting global solution of the problem. Three-dimensional spherically symmetric motion of a compressible viscous polytropic ideal gas was investigated by Itaya [6] and others (see for example, [10,14]) in annulus or external domain, Ducomet [4] in the case of gaseous star having free-boundary and rigid core in the centre. Among these works it is in [4, 5, 10, 13] that the “large” external force is considered, more precisely, the self-gravitation in [4, 5, 13], a potential force of the core in [10].

In this talk we construct the unique global classical solution of system (1.1)-(1.4) which extends the results [4] in respect of considering radiative effect and both influence of the self-gravitation and the potential force. But the difficulty of our problem is mainly caused by radiative components of equations of state and θ-dependency of the conductivity.

Let $\Omega := (0, 1)$, m be a nonnegative integer and $0 < \sigma < 1$. T is a positive constant and $Q_T := \Omega \times (0, T)$. We use the familiar notations $C^{m+\sigma}(\Omega)$, $C^{\sigma,\,\sigma/2}_{x,\,t}(Q_T)$, $C^{2+\sigma,\,1+\sigma/2}_{x,\,t}(Q_T)$ for the Hölder spaces (see for example, [9]).

Our main result is

Theorem 1 (Global Solution) *Let $\alpha \in (0, 1)$, $3 \le q < 39$ and $0 \le \beta < q + 9$. Assume that*

$$(v_0, u_0, \theta_0, z_0) \in C^{1+\alpha}(\Omega) \times \left(C^{2+\alpha}(\Omega)\right)^3$$

satisfies (1.4) and $v_0(x) > 0$, $\theta_0(x) > 0$, $0 \le z_0(x) \le 1$ for $x \in \overline{\Omega}$. Then there exists a unique solution (v, u, θ, z) of the initial-boundary value problem (1.1)-(1.3) such that for any $T > 0$

$$(v,\, v_x, v_t, u, \theta, z) \in \left(C^{\alpha,\,\alpha/2}_{x,\,t}(Q_T)\right)^3 \times \left(C^{2+\alpha,\,1+\alpha/2}_{x,\,t}(Q_T)\right)^3$$

and

$$v(x, t),\ \theta(x, t) > 0, \quad 0 \le z(x, t) \le 1 \quad \textit{for}\ \ (x, t) \in \overline{Q_T}.$$

Proof of Theorem 1 is based on the local existence theorem and a priori estimates. The fundamental theorem about the existence and the uniqueness of the local-in-time classical solution in three-dimensional free-boundary case was firstly established by Tani [12] under sufficiently general conditions. Since it is easy to verify that this manner to prove local existence theorem can be adapted without essential modification to our spherically symmetric and reacting case, to prove Theorem 1 it is sufficient to establish the following a priori boundedness.

Proposition 1 (A priori Estimates) *Let T be an arbitrary positive constant. Assume that α, q, β and the initial data satisfy the hypotheses of Theorem 1, and problem (1.1)-(1.3) has a solution (v, u, θ, z) such that*

$$(v,\, v_x, v_t, u, \theta, z) \in \left(C^{\alpha,\,\alpha/2}_{x,\,t}(Q_T)\right)^3 \times \left(C^{2+\alpha,\,1+\alpha/2}_{x,\,t}(Q_T)\right)^3.$$

Then there exists a positive constant M depending on the initial data and T such that

$$|v,\, v_x, v_t|_{\alpha,\,\alpha/2},\ |u, \theta, z|_{2+\alpha,\,1+\alpha/2} \ \le\ M,$$

$$v(x,t),\ \theta(x,t) \ge\ 1/M, \quad 0 \le z(x,t) \le 1 \quad \textit{for}\ \ (x,t) \in \overline{Q_T}.$$

2 Key Lemmas for Proving Proposition 1.

In proving Proposition 1, we need several lemmas concerning the estimates of the solution and its derivatives. The essential one of them is to get pointwise estimate of the specific volume v. To do this Kazhikhov and Shelukhin [8] firstly derived the useful representation formula of v. In the present case, we have

Lemma 1 *The identity*

$$v(x,t) = \frac{1}{\mathrm{P}(x,t)\mathrm{Q}(x,t)\mathrm{R}(x,t)} \left(v_0 + \frac{R}{\zeta}\int_0^t \theta(x,\tau)\mathrm{P}(x,\tau)\mathrm{Q}(x,\tau)\mathrm{R}(x,\tau)\,\mathrm{d}\tau \right)$$

holds, whrere

$$\begin{cases} \mathrm{P}(x,t) := \left(\dfrac{r_0(1)}{r(1,t)}\right)^{4\mu/\nu} \exp\left[\, \dfrac{1}{\zeta}\displaystyle\int_x^1 \left(\frac{u}{r^2} - \frac{u_0}{{r_0}^2}\right)\mathrm{d}\xi \,\right], \\ \mathrm{Q}(x,t) := \exp\left\{ \dfrac{p_e}{\zeta}t + \dfrac{1}{\zeta}\displaystyle\int_0^t\!\!\int_x^1 \left[\, \frac{2u^2}{r^3} + \frac{G(\xi+M_0)}{r^4} \,\right] \mathrm{d}\xi\,\mathrm{d}\tau \right\}, \\ \mathrm{R}(x,t) := \exp\left(-\dfrac{a}{3\zeta}\displaystyle\int_0^t \theta(x,\tau)^4\,\mathrm{d}\tau \right). \end{cases}$$

From this formula, we can obtain a priori bounds of v.

Lemma 2 *For any $(x,t) \in \overline{Q_T}$*

$$C_T^{-1} \le v(x,t) \le C_T, \tag{2.1}$$

where C_T is a positive constant depending on the initial data and T.

Although from the influence of radiation the estimate above is depending on T, if the external pressure p_e is sufficiently large, we can get upper and lower bounds of v uniformly in T.

References

[1] J. Bebernes and A. Bressan, Global a priori estimates for a viscous reactive gas, *Proc. Roy. Soc. Edinb.*, **101A** (1985), 321-333.

[2] S. Chandrasekhar, *An introduction to the study of stellar structure*, Dover, 1957.

[3] G. -Q. Chen, Global solution to the compressible Navier-Stokes equations for a reacting mixture, *SIAM J. Math. Anal.*, **23** (1992), 609-634.

[4] B. Ducomet, Some asymptotics for a reactive Navier-Stokes-Posson system, *Math. Models Methods Appl. Sci.*, **9** (1999), 1039-1076.

[5] B. Ducomet, Some stability results for reactive Navier-Stokes-Poisson systems, *Evol. Equ.: Existence Regularity Singularities Banach Center Publ.*, **52** (2000), 83-118.

[6] N. Itaya, A spherically symmetric global-in-time solution to the compressible Navier-Stokes equations, *in Japanese, RIMS kokyuroku, RIMS Kyoto Univ.*, **656** (1988), 51-66.

[7] S. Jiang, On initial boundary value problems for a viscous heat-conducting one-dimensional real gas, *J. Differential Equations*, **110** (1994), 157-181.

[8] A. V. Kazhikhov and V. V. Shelukhin, Unique global solution with respect to time of the initial-boundary value problems for one-dimensional equations of a viscous gas, *J. Appl. Math. Mech.*, **41** (1977), 273-282.

[9] O. A. Ladyženskaja, V. A. Solonnikov, and N. N. Ural'ceva, *Linear and quasi-linear equations of parabolic type*, Transl. Math. Monogr., vol. 23, Amer. Math. Soc., Providence, RI, 1968.

[10] T. Nakamura and S. Nishibata, Large-time behavior of spherically symmetric flow for viscous polytropic ideal gas, *preprint.*

[11] T. Nagasawa, On the outer pressure problem of the one-dimensional polytropic ideal gas, *Japan J. Appl. Math.*, **5**, (1988), 53-85.

[12] A. Tani, On the free boundary value problem for the compressible viscous fluid motion, *J. Math. Kyoto Univ.*, **21** (1981), 839-859.

[13] M. Umehara and A. Tani, Global solution to the one-dimensional equations for a self-gravitating viscous radiative and reactive gas, to appear in *J. Differential Equations.*

[14] S. Yanagi, Asymptotic stability of the spherically symmetric solutions for a viscous polytropic gas in a field of external forces, *Trans. Theory Statistical phys.*, **29** (2000), 333-353.