The local properties of critical points obtained by a mountain pass theorem on the case without the Palais-Smale condition

(峠の定理により得られる臨界点の局所的性質：

Palais-Smale 条件の無い場合について)*

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We are concerned with a local property around the critical points obtained by the mountain pass theorem under the absence of the Palais-Smale condition. We start with recalling the standard mountain pass lemma.

Let a real Banach space $(X, \|\cdot\|)$, a C^1 functional $I : X \to \mathbf{R}$, and $u_0, u_1 \in X$ be given. We take

$$\Gamma = \{\gamma \in C([0,1], X) \mid \gamma(0) = u_0,\ \gamma(1) = u_1\}$$

and call

$$c_I = \inf_{\gamma\in\Gamma} \max_{\gamma} I \tag{1}$$

the mountain pass value of I. We call the triplet (I, u_0, u_1) has the mountain pass structure if

$$c_I > \max\{I(u_0), I(u_1)\} \tag{2}$$

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holds. Next, $\{u_k\} \subset X$ is called a Palais-Smale sequence if

$$I(u_k) \to c \qquad \text{and} \qquad I'(u_k) \to 0 \text{ in } X^* \tag{3}$$

for some $c \in \mathbf{R}$ and the Palais-Smale condition, denoted by (PS), means that any Palais-Smale sequence $\{u_k\} \subset X$ admits a subsequence converging strongly in X. The following is one version of the mountain pass theorem originated with Ambrosetti-Rabinowitz [1]:

Theorem 1 ([2]) *Suppose the mountain pass structure (2) and the Palais-Smale condition (PS). Then, the mountain pass value c_I defined by (1) is a critical value of I, i.e., there is $v \in X$ satisfying $I'(v) = 0$ and $I(v) = c_I$.*

The Palais-Smale condition (PS) is known to be weakened to $(PS)_{c_I}$, which means that any sequence $\{u_k\} \subset X$ satisfying (3) with $c = c_I$ has a strongly converging subsequence. See, e.g., [7], p.101. However, lack of these conditions is observed in many cases. Among them is the functional

$$I_\lambda(u) = \frac{1}{2}\int_M |\nabla u|^2 - \lambda \log\left(\frac{1}{|M|}\int_M e^u\right)$$

defined for $u \in E$, where (M, g) is a compact orientable Riemannian manifold without boundary and

$$E = \left\{ u \in H^1(M) \mid \int_M u = 0 \right\}.$$

Its Euler-Lagrange equation

$$-\Delta_g v = \lambda\left(\frac{e^v}{\int_\Omega e^v} - \frac{1}{|M|}\right), \qquad \int_M v = 0 \tag{4}$$

is called the mean field equation studied by many authors. In this example, lack of the Palais-Smale condition arises because the Palais-Smale sequence is not necessarily bounded when $\lambda \geq 8\pi$. See [5, 6] and the references therein for the"bubbling" of such sequences.

In this problem, the trivial solution $v = 0$ is linearly stable for $\lambda < \mu_1 |M|$ and I_λ is unbounded for $\lambda > 8\pi$, where μ_1 is the first eigenvalue of $-\Delta_g$, e.g., $\mu_1 = 4\pi^2$ if M is a flat torus with the fundamental cell domain $[0,1] \times [0,1]$. Thus, we obtain the mountain pass structure in $8\pi < \lambda < \mu_1 |M|$, i.e., it holds that (2) for $v_0 = 0$ and $v_1 \neq 0$ with $\|v_1\|_E \gg 1$. Moreover,

$$\log\left(\frac{1}{|M|}\int_\Omega e^v\right) \geq \log e^{\frac{1}{|M|}\int_M v} = 0$$

by Jensen's inequality, and therefore, $I_\lambda(u)$ is a non-increasing function of λ for fixed u. This provides $(I_\lambda, 0, v_1)$ for some fixed $v_1 \in E$ with the mountain pass structure for every λ in the fixed interval $[\lambda_0, \mu_1|M|) \subset (8\pi, \mu_1|M|)$. Consequently we are able to assume $c(\lambda) \equiv c_{I_\lambda}$ is a non-increasing function of λ and

$$\frac{d}{d\lambda}c(\lambda)(\equiv c'(\lambda))$$

exists for a.e. λ, which guarantees a bounded $(PS)_{c(\lambda)}$ sequence when $c'(\lambda)$ exists. These arguments are sometimes called the Struwe's monotonicity trick. In spite of the lack of the Palais-Smale condition, we still have an analogous result to Theorem 1 by this monotonicity trick.

Theorem 2 ([8]) *For a.e. $\lambda \in (8\pi, \mu_1 |M|)$, the mountain pass value*

$$c(\lambda)(\equiv c_{I_\lambda})$$

is a critical value of the above defined I_λ.

Concerning the existence of the non-trivial solution, the above residual set of λ is compensated by the blowup analysis, i.e., the quantized blowup mechanism of the solution sequence [4]. More precisely, any $\lambda \in (8\pi, \mu_1 |M|) \setminus 8\pi\mathbf{N}$ admits a non-trivial solution to (4) by this mechanism.

There is, on the other hand, study on the local structure of the mountain pass critical point.

Definition 3 ([2]) *Given a critical point v of $I \in C^1(X, \mathbf{R})$, we say the following:*

1. *It is a local minimum if there is an open neighbourhood of v, denoted by V such that $I(u) \geq I(v)$ for any $u \in V$.*
2. *It is mountain pass type if any open neighbourhood U of v has the properties that $U \cap I^d \neq \emptyset$ and $U \cap I^d$ is not path-connected, where $d = I(v)$ and*
$$I^d = \{u \in X \mid I(u) < d\}.$$

Theorem 4 ([2]) *The critical value in Theorem 1 can be either a local minimum or mountain pass type.*

The proof of the above theorem also uses the Palais-Smale condition, more precisely, $(PS)_{I_c}$. The purpose of this talk is to present that Theorems 2 and 4 are compatible.

Theorem 5 (Main Theorem) *In Theorem 2, if $c'(\lambda)$ exists and $\lambda \notin 8\pi\mathbf{N}$, then there is a local minimum or a mountain pass type critical point with the critical value $c(\lambda)$.*

We use the argument of [3] developed for the proof of Theorem 2.

References

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