The distribution of eigenfunctions in the Anderson model

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Abstract

We study the distribution of eigenvalues and corresponding eigenvalues in the localized region of the Anderson model. We show (1) all eigenfunctions in the localized region I are uniformly distributed, (2) distributions of the eigenfunctions whose eigenvalues are in the order of L^{-d} obeys a Poissonian law for large L, (3) eigenfunctions whose eigenvalues are in the order of L^{-2d} are repulsive each other.

1 Introduction

The Anderson model on $l^2(\mathbf{Z}^d)$ is given by

$$(H\phi)(x) = \sum_{|x-y|=1} \phi(y) + \lambda V_\omega(x)\phi(x), \quad \phi \in l^2(\mathbf{Z}^d)$$

where $\lambda \neq 0$ is the coupling constant and $\{V_\omega(x)\}_{x\in\mathbf{Z}^d}$ are the independent, identically distributed real-valued random variables on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ such that the common distribution has a bounded density ρ. The following results are well-known.
(1) the spectrum of H is almost surely equal to a fixed set $\Sigma(\subset \mathbf{R})$ [8]:

$$\sigma(H) = \Sigma \quad a.s., \quad \Sigma := [-2d, 2d] + \lambda \text{ supp } \rho.$$

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(2) (Anderson localization, [1, 2, 4, 6, 12]) There exists an interval $I(\subset \Sigma)$ such that, with probability 1, $\sigma(H)\cap I$ is dense pure point with exponentially decaying eigenfuntions. I can typically be taken (i) $I = \Sigma$ if $\lambda \gg 1$, (ii) in the extreme energy, (iii) in the band edges, (iv) away from the spectrum of the free Laplacian if $\lambda \ll 1$.

The aim of this talk is to study the distribution of eigenvalues and eigenvalues where the Anderson localization takes place. We first fix notations.

(1) $\Lambda_L(x)$ is the finite box with center $x \in \mathbf{Z}^d$ and size $L > 0$ and $\partial\Lambda$ is the "boundary" of the box Λ :

$$\Lambda_L(x) := \{y \in \mathbf{Z}^d : |y_j - x_j| \le \frac{L}{2}, j = 1, 2, \cdots, d\}$$
$$\partial\Lambda := \{x \in \Lambda : |y - x| = 1, \quad \text{for some } y \notin \Lambda\}$$

(2) For a box Λ, let $H_\Lambda := H|_\Lambda$ is the restriction of H on Λ.
(3) Let $\gamma > 0, E \in \mathbf{R}$ and let $G_\Lambda(E; x, y) := \langle x|(H_\Lambda - E)^{-1}|y\rangle$. We say the box $\Lambda_L(x)$ is (γ, E)-regular iff $E \notin \sigma(H_{\Lambda_L(x)})$ and

$$\sum_{y \in \partial\Lambda_L(x)} |G_{\Lambda_L(x)}(E; x, y)| \le e^{-\gamma \frac{L}{2}}.$$

(4) For $\phi \in l^2(\mathbf{Z}^d)$, let $X(\phi)$ be the set of its center of localization :

$$X(\phi) := \{x \in \mathbf{Z}^d : |\phi(x)| = \max_{y \in \mathbf{Z}^d} |\phi(y)|\}$$

This definition is due to [5]. We choose $x(\phi) \in X(\phi)$ according to an order on $\mathbf{Z}^d$. For an eigenvalue E of H, we choose the corresponding eigenfunction ϕ_E (according to some procedure) and set $X(E) = X(\phi), x(E) \in X(E)$.
(5) Let ν be the density of states measure on $\mathbf{R}$:

$$\nu(A) := \mathbf{E}[\langle 0|P_A(H)|0\rangle], \quad A \in \mathcal{B}(\mathbf{R})$$

where $P_A(H)$ is the spectral projection of H corresponding to A.

2 Results

Throughout this talk, we assume :

Assumption *We can find an interval $I(\subset \Sigma)$, $p > 6d$ such that*

$$\mathbf{P}\left(\textit{For any } E \in I,\ H_{\Lambda_{L_0}(0)} \textit{ is } (\gamma, E)\textit{-regular} \right) \geq 1 - L_0^{-p}$$

for sufficiently large L_0.

This assumption is known to hold in some regions in Σ where Anderson localization holds, whose location is mentioned in Introduction. Take any α with $1 < \alpha < \frac{2p}{p+2d}$ and set

$$L_{k+1} := L_k^{\alpha}, \quad \Lambda_k(x) := \Lambda_{L_k}(x), \quad k = 0, 1, \cdots.$$

Then by the multiscale analysis, we have the following estimate by which we deduce the exponential decay of eigenfunctions [12] :

$$\mathbf{P}\left(\text{ For any } E \in I, \text{ either } \Lambda_k(x) \text{ or } \Lambda_k(y) \text{ is } (\gamma, E)\text{-regular } \right) \geq 1 - L_k^{-2p}$$

for any $x, y \in \mathbf{Z}^d$ with $|x - y| > L_k$. Let $\Lambda_k = \{1, 2, \cdots, L_k\}^d$, $k = 1, 2, \cdots$ be a box of size L_k and let $H_k := H|_{\Lambda_k}$ with periodic boundary condition.

(1) **Macroscopic Limit**
Let $\{E_j(\Lambda_k)\}$ be the eigenvalues of H_k and let $\{F_j(\Lambda_k)\} = \{E_j(\Lambda_k)\} \cap I$. We consider the following random measure on $I \times K$ ($K := [0, 1]^d$).

$$\xi_k := \frac{1}{|\Lambda_k|} \sum_j \delta_{X_j}, \quad X_j := (F_j(\Lambda_k), L_k^{-1} x(F_j(\Lambda_k))) \in I \times K.$$

Theorem 2.1 *$\xi_k \stackrel{v}{\to} \nu \otimes dx$, a.s. as $k \to \infty$.*

Theorem 2.1 roughly says that the center of localizations are uniformly distributed. Theorem 2.1 holds for most random models for which the multiscale analysis is applicable(e.g., [3, 7]).

(2) **Local fluctuation**
Since eigenvalues of H_Λ typically arranges in the order of $|\Lambda|$, we take a reference energy $E_0 \in I$, consider the scaled eigenvalues, and define the following point process on $\mathbf{R} \times K$.

$$\xi'_k := \sum_j \delta_{Y_j}, \quad Y_j := (|\Lambda_k|(E_j(\Lambda_k) - E_0), L_k^{-1} x(E_j(\Lambda_k))) \in \mathbf{R} \times K.$$

Theorem 2.2 *Suppose* $E_0 \in I$ *is a Lebesgue point of* ν. *Then* $\xi'_k \xrightarrow{d} \zeta_P$ *as* $k \to \infty$ *where* ζ_P *is the Poisson process on* $\mathbf{R} \times K$ *with intensity measure* $\mathbf{E}\zeta_P = \frac{d\nu}{dE}(E_0)dE \times dx$.

For its proof, Minami's estimate [10] is the essential ingredient. For other models where Anderson localization is known to hold [3, 7], we can show that the limiting points of $\{\xi'_k\}$ are infinitely divisible with absolutely continuous intensity measure. Theorems 2.1, 2.2 can be stated in another form [9].

(3) **Repulsion of eigenfunctions**

We consider eigenvalues which are much closer than $|\Lambda_k|^{-1}$ and see that the corresponding eigenfunctions are repulsive.

Theorem 2.3 *[11] Let* $d_k := |\Lambda_k|^{-2}k^{-2}$. *Then for a.e.* ω *and for any eigenvalue* E *of* H, *we can find* $k_0 = k_0(E, \omega)$ *such that for* $k \geq k_0$, *any other eigenvalues* E' *with* $|E - E'| \leq d_k$ *satisfy* $|x(E) - x(E')| \geq L_k$.

Theorem 2.3 roughly says that, if two eigenvalues E, E' of H satisfy $|E - E'| \leq L^{-2d}$, then the corresponding centers of localization must satisfy $|x(E) - x(E')| \geq L$.

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