３つの δ 関数を初期データに持つ非線形シュレーディンガー方程式の解の時間大域的評価について

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1 Introduction and Main Results

We consider the initial value problem of the nonlinear Schrödinger equation like

$$(1.1)\qquad \begin{cases} i\partial_t u = -\Delta u + \lambda \mathcal{N}(u) \\ u(0,x) = \mu_{00}\delta_0 + \mu_{10}\delta_a + \mu_{01}\delta_b \end{cases}$$

where $(t,x) \in \mathbf{R} \times \mathbf{R}^n$ $(n \geq 1)$, $\partial_t = \partial/\partial t$ and $\Delta = \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2 + \cdots + \partial^2/\partial x_n^2$. The unknown variable $u = u(t,x)$ takes a complex number. The nonlinearity $\mathcal{N}(u)$ is of the gauge invariant power type given by

$$\mathcal{N}(u) = |u|^{p-1}u \quad \text{with } 1 < p < 1 + 2/n.$$

The nonlinear coefficient λ belongs to $\mathbf{C}$ (the set of complex numbers). In particular, if $\mathrm{Im}\lambda < 0$, the nonlinear term causes dissipative effect. In the initial data, δ_a denotes the well-known point mass measure supported at $x = a \in \mathbf{R}^n$ and μ_{jk} $(j,k = 0,1)$ are complex numbers.

About (1.1), the speaker showed that

- If $u(0,x) = \mu\delta_0$, then $u(t,x) = A(t)U(t)\delta_0$, where $U(t) = \exp(it\Delta)$ and $A(t)$ depends only on time variable t. Note that $A(t)$ blows up at $t = T^* > 0$ if $\mathrm{Im}\lambda > 0$ and globally exists if $\mathrm{Im}\lambda \leq 0$.
- If $u(0,x) = \mu_0\delta_0 + \mu_1\delta_a$, then $u(t,x) = \sum_{j\in\mathbf{Z}} A_j(t)U(t)\delta_{ja}$. Note that, roughly speaking, $A_j(t)$ blow up at $t = T^* > 0$ if $\mathrm{Im}\lambda > 0$ and globally exist if $\mathrm{Im}\lambda \leq 0$.

Hence our present concern is to consider the triple δ-function case. If $a = qb$ for some $q \in \mathbf{Q}$ ($\mathbf{Q}$ denotes the quotient number field), then the δ-functions are located at three points on the 1-dimensional lattice and (1.1) is solvable globaly in time — the proof follows similarly to the double δ-function case. Therefore, in what follows, we restrict ourselvs to observing the case $a \neq qb$ for any $q \in \mathbf{Q}$. Before stating the time local result, let us introduce several notations. The weighted sequence space $\ell^2_\alpha(\mathbf{Z}^2)$ is defined by

$$\ell^2_\alpha(\mathbf{Z}^2) = \{\{A_{jk}\}_{j,k\in\mathbf{Z}};\ \|\{A_{jk}\}_{j,k\in\mathbf{Z}}\|_{\ell^2_\alpha(\mathbf{Z}^2)} < \infty\},$$

where $\|\{A_{jk}\}_{j,k\in\mathbf{Z}}\|^2_{\ell^2_\alpha(\mathbf{Z}^2)} = \sum_{j,k\in\mathbf{Z}}(1+|j|+|k|)^{2\alpha}|A_{jk}|^2$. For simplicity of the description, we often use $\{A_{jk}\}$ in place of $\{A_{jk}\}_{j,k\in\mathbf{Z}}$. Then the time local result is

Theorem 1.1 (local result) *Let $\lambda \in \mathbf{C}$ and $1 < \alpha < p$. Then, for some $T > 0$, there exists a unique solution to (1.1) described as*

$$u(t,x) = \sum_{j,k\in\mathbf{Z}} A_{jk}(t)U(t)\delta_{ja+kb}, \tag{1.2}$$

where $\{A_{jk}(t)\} \in C([0,T];\ell^2_\alpha(\mathbf{Z}^2)) \cap C^1((0,T];\ell^2_\alpha(\mathbf{Z}^2))$ with $A_{jk}(0) = \mu_{jk}$ if $(j,k) = (0,0),(1,0),(0,1)$ and $A_{jk}(0) = 0$ otherwise.

Remark 1.1. The solution in Theorem 1.1 also causes the generation of new modes. Note that, for $t \neq 0$, $U(-t)u(t)$ looks like a point mass measure supported at 2-dimensional lattice points if $a \not\parallel b$, and densely distributed on the line along vector a if $a \parallel b$ and $a \neq qb$ for any $q \in \mathbf{Q}$. Reading the proof of Theorem 1.1, we see that it is possible to construct a solution even when the initial data consists of infintely many δ-functions such as $u(0,x) = \sum_{j,k\in\mathbf{Z}} \mu_{jk}\delta_{ja+kb}$ with $\{\mu_{jk}\} \in \ell^2_\alpha(\mathbf{Z}^2)$ and $\alpha > 1$.

The sign of Imλ determines the blowing-up or global existence of the solution.

Theorem 1.2 (blowing-up result) *Let Im$\lambda > 0$. Then, the solution in Theorem 1.1 blows up in positive finite time. Precisely speaking, $\lim_{t\uparrow T^*}\|\{A_{jk}(t)\}\|_{\ell^2_0(\mathbf{Z}^2)} = \infty$ for some $T^* > 0$.*

As for the global existence, the difficulty largely depends on whether a and b are parallel or not, which does not arise in the single and double δ-function case.

Theorem 1.3 (global result) *(1) Let $a \not\parallel b$. Then, if Im$\lambda \leq 0$, there exists a unique global solution to (1.1) described as in Theorem 1.1, where $\{A_{jk}(t)\} \in C([0,\infty);\ell^2_\alpha(\mathbf{Z}^2)) \cap C^1((0,\infty);\ell^2_\alpha(\mathbf{Z}^2))$.*

(2) Let $a \parallel b$ and $a \neq qb$ for any $q \in \mathbf{Q}$. Then, if Im$\lambda \leq 0$ and additionally $|\text{Re }\lambda| \leq \dfrac{2\sqrt{p}}{p-1}|\text{Im }\lambda|$, there exists a unique global solution to (1.1) described as in Theorem 1.1, where $\{A_{jk}(t)\} \in C([0,\infty);\ell^2_\alpha(\mathbf{Z}^2)) \cap C^1((0,\infty);\ell^2_\alpha(\mathbf{Z}^2))$.

Remark 1.2. When $a \not\parallel b$, the important matter is the equivalence of $\|\{(ja+kb)A_{jk}\}\|_{\ell^2_0(\mathbf{Z}^2)}$ and $\|\{jA_{jk}\}\|_{\ell^2_0(\mathbf{Z}^2)} + \|\{kA_{jk}\}\|_{\ell^2_0(\mathbf{Z}^2)}$. However, this is not the case if $a \parallel b$. As for Theorem 1.3 (2), it is still open whether the additional condition $|\text{Re}\lambda| \leq \dfrac{2\sqrt{p}}{p-1}|\text{Im}\lambda|$ is removed or not. In our proof, this condition will be applied to obtain the time global estimate of

$\|\{A_{jk}(t)\}\|_{\ell_1^2(\mathbf{Z}^2)}$ (This gives a rise to the desired estimate in $\ell_\alpha^2(\mathbf{Z}^2)$). The key to derive this esimate is Liskevich-Perelmuter's inequality [5], i.e., if $\mathrm{Im}\lambda \le 0$ and $|\mathrm{Re}\lambda| \le \dfrac{2\sqrt{p}}{p-1}|\mathrm{Im}\lambda|$, then it follows that $\mathrm{Im}\left(\lambda(\mathcal{N}(v_1)-\mathcal{N}(v_2))\overline{(v_1-v_2)}\right) \le 0$.

We close this abstract by giving some more notations used in this talk. Let $\mathbf{T} = \mathbf{R}/2\pi\mathbf{Z}$ where $\mathbf{Z}$ stands for the integer set. The quantity $\|f\|_{L^q(\mathbf{T}^2)}$ denotes $\left(\int_{\mathbf{T}^2} |f(\theta_1,\theta_2)|^q \, d\theta_1 d\theta_2\right)^{1/q}$. We next define the Besov space for periodic functions. Let $[s]$ be the greatest integer not exceeding s. Then, if s is not integer and $1 < q, r < \infty$, the Besov space $B^s_{q,r}(\mathbf{T}^2)$ is defined by

$$B^s_{q,r}(\mathbf{T}^2) = \{f \in L^q(\mathbf{T}^2);\ \|f\|_{B^s_{q,r}(\mathbf{T}^2)} < \infty\},$$

where

$$\begin{aligned}
\|f\|_{B^s_{q,r}(\mathbf{T}^2)} &\equiv \|f\|_{L^q(\mathbf{T}^2)} + \|f\|_{\dot{B}^s_{q,r}} \\
&\equiv \|f\|_{L^q(\mathbf{T}^2)} + \left(\int_0^\infty \tau^{-rs-1} \sup_{|h|<\tau} \|d_h^{[s]+1} f\|^r_{L^q(\mathbf{T}^2)} \, d\tau\right)^{1/r}
\end{aligned}$$

with $h = (h_1, h_2)$ and $d_h^N f(\theta_1,\theta_2) = \sum_{j=0}^{N} \begin{pmatrix} N \\ j \end{pmatrix} (-1)^k f(\theta_1 + jh_1, \theta_2 + jh_2)$. We remark that, if $0 \le \sigma \le 1$ and $1/q = \sigma/q_1 + (1-\sigma)/q_0$ with $1 \le q_1, q_0 \le \infty$, then the Gagliardo-Nirenberg type inequality $\|f\|_{\dot{B}^{\sigma s}_{q,r/\sigma}(\mathbf{T}^2)} \le C\|f\|^{\sigma}_{\dot{B}^s_{q_1,r}(\mathbf{T}^2)}\|f\|^{1-\sigma}_{L^{q_0}(\mathbf{T}^2)}$ follows from the above definition. We also note that $\|f\|_{B^s_{2,2}(\mathbf{T}^2)}$ is equivalent to

$$\|f\|_{H^s(\mathbf{T}^2)} \equiv \left(\sum_{j,k\in\mathbf{Z}} (1+|j|+|k|)^{2s}|C_{jk}|^2\right)^{1/2},$$

where C_{jk} is the Fourier coefficient of f given by $(2\pi)^{-2}\int_{\mathbf{T}^2} f(\theta_1,\theta_2)e^{-i(j\theta_1+k\theta_2)} \, d\theta_1 d\theta_2$. Also, the inner product of $f(\theta_1,\theta_2)$ and $g(\theta_1,\theta_2) \in L^2(\mathbf{T}^2)$ is defined by $\langle f, g\rangle_{\theta_1,\theta_2} = \int_{\mathbf{T}^2} f(\theta_1,\theta_2)\overline{g(\theta_1,\theta_2)} \, d\theta_1 d\theta_2$.

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