Entire solutions to the generalized parabolic k-Hessian equation

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This talk is based on a joint work with Saori Nakamori (Hiroshima University).

About a hundred years ago, Bernstein [2] proved the following theorem.

Theorem 1. [2] *If $f \in C^2(\mathbb{R}^2)$ and the graph of $z = z(x, y)$ is a minimal surface in $\mathbb{R}^3$, then f is necessarily an affine function of x and y.*

This theorem gives the characterization of entire solutions to the minimal surface equation defined in the whole plane $\mathbb{R}^2$.

Many problems on the classification of entire solutions to PDEs have been extensively studied. We list some results concerning Bernstein type theorems for *fully nonlinear* equations. First, for Monge-Ampère equation, the following theorem is known.

Theorem 2. *Let $u \in C^4(\mathbb{R}^n)$ be a convex solution to*

$$\det D^2 u = 1 \quad \text{in } \mathbb{R}^n.$$

Then u is a quadratic polynomial.

Theorem 2 was proved by Jörgens [6] for $n = 2$, by Calabi [3] for $n \leq 5$, and by Pogorelov [9] for arbitrary $n \geq 2$ (see also [4] for a simpler proof).

Later, Bao, Chen, Guan and Ji [1] extended this result to the so-called *k-Hessian equation* of the form

$$F_k(D^2 u) = S_k(\lambda_1, \ldots, \lambda_n) = 1 \quad \text{in } \mathbb{R}^n, \tag{1}$$

for $1 \leq k \leq n$. Here, for a C^2 function u, $\lambda_1, \ldots, \lambda_n$ denote the eigenvalues of the Hessian matrix $D^2 u$, and S_k denotes the k-th elementary symmetric function, that is

$$S_k(\lambda_1, \ldots, \lambda_n) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \lambda_{i_1} \cdots \lambda_{i_k}. \tag{2}$$

We note that Laplace operator Δu and Monge-Ampère operator $\det D^2 u$ correspond respectively to the special cases $k = 1$ and $k = n$ in (2).

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Theorem 3. [1] *Let $1 \le k \le n$ and $u \in C^4(\mathbb{R}^n)$ be a strictly convex solution to* (1). *Suppose that there exist constants $A, B > 0$ such that for all $x \in \mathbb{R}^n$,*

$$u(x) \ge A|x|^2 - B.$$

Then u is a quadratic polynomial.

Next, Gutiérrez and Huang [5] extended Theorem 2 to the parabolic analogue of Monge-Ampère equation

$$-u_t \det D^2 u = 1 \quad \text{in } \mathbb{R}^n \times (-\infty, 0]. \tag{3}$$

Here a function $u = u(x,t) : \mathbb{R}^n \times (-\infty, 0] \to \mathbb{R}$ is said to be *convex-monotone* (resp. *strictly convex-monotone*) if it is convex (resp. strictly convex) in x and non-increasing (resp. decreasing) in t.

Theorem 4. [5] *Let $u \in C^{4,2}(\mathbb{R}^n \times (-\infty, 0])$ be a convex-monotone solution to* (3). *Suppose that there exist constants $m_1 \ge m_2 > 0$ such that for all $(x,t) \in \mathbb{R}^n \times (-\infty, 0]$,*

$$-m_1 \le u_t(x,t) \le -m_2. \tag{4}$$

Then u has the form $u(x,t) = -C_1 t + p(x)$ where $C_1 > 0$ is a constant and p is a quadratic polynomial.

We note that Xiong and Bao [10] have recently obtained Bernstein type theorems for more general parabolic Monge-Ampère equations, such as $u_t = (\det D^2 u)^{1/n}$ and $u_t = \log \det D^2 u$. However, as far as we know, Bernstein type theorems for parabolic fully nonlinear equations are known only for the parabolic Monge-Ampère type equations.

In this talk, we are concerned with the parabolic analogue of k-Hessian equation of the following form

$$u_t = \mu\left(F_k(D^2u)^{\frac{1}{k}}\right) \quad \text{in } \mathbb{R}^n \times (-\infty, 0], \tag{5}$$

where $\mu : (0, \infty) \to \mathbb{R}$ is a function. Here is a main result of this talk.

Theorem 5. [8] *Let $\mu \in C^2(0,\infty)$, $1 \le k \le n$ and $u \in C^{4,2}(\mathbb{R}^n \times (-\infty, 0])$ be a strictly convex-monotone solution to* (5). *Suppose that there exist constants $m_1 \ge m_2 > 0$ such that for all $(x,t) \in \mathbb{R}^n \times (-\infty, 0]$,*

$$-m_1 \le u_t(x,t) \le -m_2, \tag{6}$$

and that there exist constants $A, B > 0$ such that for all $x \in \mathbb{R}^n$,

$$u(x,0) \ge A|x|^2 - B. \tag{7}$$

Moreover, suppose that for all $s \in (0,\infty)$,

$$\mu'(s) > 0, \quad \mu''(s) \le 0, \tag{8}$$

and that

$$\mu^{-1}([-m_1, -m_2]) = [r_1, r_2] \tag{9}$$

for some positive constants r_1, r_2*, where* m_1 *and* m_2 *are constants appeared in* (6)*.*

Then, u *has the form* $u(x,t) = -mt + p(x)$ *where* $m > 0$ *is a constant and* p *is a quadratic polynomial.*

Using Theorem 5, one can obtain Bernstein type theorems for various equations including the following examples:

Example 1. (i) $-u_t F_k(D^2u) = 1$ in $\mathbb{R}^n \times (-\infty, 0]$, which was obtained by Nakamori and Takimoto [7] previously, if we set $\mu(s) = -s^{-k}$.

(ii) $-u_t F_k(D^2u)^{\frac{1}{k}} = 1$ in $\mathbb{R}^n \times (-\infty, 0]$, if we set $\mu(s) = -1/s$.

(iii) $u_t = \log F_k(D^2u)$ in $\mathbb{R}^n \times (-\infty, 0]$, if we set $\mu(s) = k \log s$.

(iv) We can also obtain Bernstein type theorem for

$$u_t = F_k(D^2u)^{\frac{1}{k}} \quad \text{in } \mathbb{R}^n \times (-\infty, 0]. \tag{10}$$

We remark that for $k = 1$, (10) reduces to the heat equation $u_t = \Delta u$ which is well-known.

The sketch of the proof of Theorem 5 and some open problems are given in the talk.

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