Motion of a Vortex Filament in an External Flow

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We consider a nonlinear model equation describing the motion of a vortex filament immersed in an incompressible and inviscid fluid. In the present problem setting, we also take into account the effect of external flow. We prove the unique solvability, locally in time, of an initial value problem posed on the one dimensional torus. The problem describes the motion of a closed vortex filament.

A vortex filament is a space curve on which the vorticity of the fluid is concentrated. Vortex filaments are used to model very thin vortex structures such as vortices that trail off airplane wings or propellers. In this talk, we prove the solvability of the following initial value problem which describes the motion of a closed vortex filament.

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(1) \qquad \begin{cases} \boldsymbol{x}_t = \dfrac{\boldsymbol{x}_s \times \boldsymbol{x}_{ss}}{|\boldsymbol{x}_s|^3} + \boldsymbol{F}(\boldsymbol{x}, t), & s \in \mathbf{T},\ t > 0, \\ \boldsymbol{x}(s, 0) = \boldsymbol{x}_0(s), & s \in \mathbf{T}, \end{cases}
$$

where $\boldsymbol{x}(s,t) = (x_1(s,t), x_2(s,t), x_3(s,t))$ is the position vector of the vortex filament parametrized by s at time t, the symbol $\times$ is the exterior product in the three dimensional Euclidean space, $\boldsymbol{F}(\cdot, t)$ is a given external flow field, $\mathbf{T}$ is the one dimensional torus $\mathbf{R}/\mathbf{Z}$, and subscripts are differentiations with the respective variables. Problem (1) describes the motion of a closed vortex filament under the influence of external flow. Such a setting can be seen as an idealization of the motion of a bubblering in water, where the thickness of the ring is taken to be zero and some environmental flow is also present. The main result we present in this talk is as follows.

Theorem 1 *For $T > 0$ and a natural number $m \geq 4$, if the initial filament $\boldsymbol{x}_0$ satisfies $\boldsymbol{x}_0 \in H^m(\mathbf{T})$ and $|\boldsymbol{x}_{0s}| \equiv 1$, and the external flow $\boldsymbol{F}$ satisfies $\boldsymbol{F} \in C\big([0,T]; W^{m,\infty}(\mathbf{R}^3)\big)$, then there exists $T_0 \in (0, T]$ such that a unique solution $\boldsymbol{x}(s,t)$ of (1) exists and satisfies*

$$
\boldsymbol{x} \in C\big([0,T_0]; H^m(\mathbf{T})\big) \cap C^1\big([0,T_0]; H^{m-2}(\mathbf{T})\big)
$$

The above theorem gives the time-local unique solvability of (1). We note that Nishiyama [1] proved the existence of a solution in $C\big([0,T]; H^2(\mathbf{T})\big)$ for any $T > 0$, and hence the above theorem extends his result to higher order Sobolev spaces and also ensures the uniqueness of the solution.

In this talk, the outline of the proof of Theorem 1 will be given, focusing only on the crucial parts.

References

[1] Nishiyama, *On the motion of a vortex filament in an external flow according to the localized induction approximation*, Proc. Roy. Soc. Edinburgh Sect. A, **129** (1999), no. 3, pp. 617–626.