On the effect of spatial expansion
on nonlinear Schrödinger equations

中村　誠　(山形大学・理学部)

2015 年 3 月 14 日　於熊本大学

1 Introduction

We consider local and global solutions for the Cauchy problem of nonlinear Schrödinger equations derived from the nonrelativistic limit of nonlinear Klein-Gordon equations in de Sitter spacetime. We put

spatial dimension : $n \geq 1$	Planck constant : $\hbar := h/2\pi$
mass : $m > 0$	Hubble constant : $H \in \mathbb{R}$
weight function $b(s) := 1 - 2mHs/\hbar$	$S_0 := \hbar/2mH$ if $H > 0$, $S_0 := \infty$ if $H \leq 0$.

For $0 \leq \mu_0 \leq n/2$ and $0 \leq S \leq S_0$, we consider

$$\begin{cases} i\dfrac{\partial u}{\partial s}(s,x) \pm \dfrac{1}{2}\,\Delta u(s,x) \mp \dfrac{V'\left(u(s,x)b(s)^{n/4}\right)}{2b(s)^{1+n/4}} = 0, \\ u(0,\cdot) = u_0(\cdot) \in H^{\mu_0}(\mathbb{R}^n) \end{cases} \tag{1.1}$$

for $(s,x) \in [0,S) \times \mathbb{R}^n$, where V' is a nonlinear function, $\Delta := \sum_{j=1}^n \partial^2/\partial x_j^2$. We say that u is a global solution of (1.1) if it exists on $[0,S_0)$.

We consider the potential of power type given by

$$V(v) := \frac{\kappa|v|^{p+1}}{p+1}, \qquad V'(v) = \kappa|v|^{p-1}v, \tag{1.2}$$

where $\kappa \in \mathbb{C}$ and $1 \leq p < \infty$. Then (1.1) is rewritten as

$$\begin{cases} i\dfrac{\partial u}{\partial s}(s,x) \pm \dfrac{1}{2}\,\Delta u(s,x) \mp \dfrac{\kappa}{2}b(s)^{n(p-1)/4-1}|u(s,x)|^{p-1}u(s,x) = 0, \\ u(0,\cdot) = u_0(\cdot) \in H^{\mu_0}(\mathbb{R}^n) \end{cases} \tag{1.3}$$

for $(s,x) \in [0,S) \times \mathbb{R}^n$ and $0 \leq \mu_0 < n/2$. The scaling number of p for the Minkowski spacetime $H = 0$ is given by $p = p(\mu_0) := 1 + 4/(n - 2\mu_0)$.

For any real numbers $2 \le q, r \le \infty$, we say that the pair (q,r) is admissible if it satisfies $1/r + 2/nq = 1/2$ and $(q,r,n) \neq (2,\infty,2)$. For $\mu_0 \ge 0$ and two admissible pairs $\{(q_j, r_j)\}_{j=1,2}$, we define

$$X^{\mu_0}([0,S)) := \{u \in C([0,S), H^{\mu_0}(\mathbb{R}^n)); \ \max_{\mu=0,\mu_0} \|u\|_{X^\mu([0,S))} < \infty\},$$

where

$$\|u\|_{X^\mu([0,S))} := \begin{cases} \|u\|_{L^\infty((0,S),L^2(\mathbb{R}^n)) \cap \bigcap_{j=1,2} L^{q_j}((0,S),L^{r_j}(\mathbb{R}^n))} & \text{if } \mu = 0, \\ \|u\|_{L^\infty((0,S),\dot{H}^\mu(\mathbb{R}^n)) \cap \bigcap_{j=1,2} L^{q_j}((0,S),\dot{B}^\mu_{r_j 2}(\mathbb{R}^n))} & \text{if } \mu > 0. \end{cases}$$

Theorem 1.1. *Let $n \ge 1$, $H \in \mathbb{R}$, $\kappa \in \mathbb{C}$, $0 \le \mu_0 < n/2$, and $1 \le p \le p(\mu_0) := 1 + 4/(n - 2\mu_0)$. Assume $\mu_0 < p$ if p is not an odd number. There exist two admissible pairs $\{(q_j, r_j)\}_{j=1,2}$ with the following properties.*

(1) (Local solutions.) For any initial data $u_0 \in H^{\mu_0}(\mathbb{R}^n)$, there exist $S > 0$ with $S \le S_0$ and a unique time local solution u of (1.3) in $X^{\mu_0}([0,S))$. Here, S depends on the norm of $\|u_0\|_{\dot{H}^{\mu_0}(\mathbb{R}^n)}$ when $p < p(\mu_0)$, and the profile of u_0 when $p = p(\mu_0)$.

(2) (Small global solutions.) Let $H \ge 0$. Let $p = p(\mu_0)$ with $\mu_0 \ge 0$ and $H \ge 0$, or let $1 < p < p(\mu_0)$ with $\mu_0 > 0$ and $H > 0$. If $\|u_0\|_{\dot{H}^{\mu_0}(\mathbb{R}^n)}$ is sufficiently small, then the solution u obtained in (1) is a global solution, namely, $S = S_0$. And u behaves as the free solution asymptotically.

Corollary 1.2. *Let $\kappa > 0$, $\mu_0 = 1$. Let H and p satisfy $H(p - 1 - 4/n) \ge 0$ and $p < 1 + 4/(n-2)$. For any data $u_0 \in H^1(\mathbb{R}^n)$, the local solution u given by (1) in Theorem 1.1 is a global solution.*

Corollary 1.3. *Let $\kappa < 0$, $\mu_0 = 1$.*

(1) Let $H \ge 0$, and $p \le 1 + 4/n$. For any data $u_0 \in H^1(\mathbb{R}^n)$, the local solution u given by (1) in Theorem 1.1 is a global solution, where we assume that $\|u_0\|_{L^2(\mathbb{R}^n)}$ is sufficiently small when $p = 1 + 4/n$.

(2) Let $H \le 0$, and $p \ge 1 + 4/n$. For any radially symmetric data $u_0 \in H^1(\mathbb{R}^n)$ with $\||x|u_0(x)\|_{L^2(\mathbb{R}^n)} < \infty$, and

$$\int_{\mathbb{R}^n} \frac{1}{2}|\nabla u_0(x)|^2 + \frac{\kappa |u_0(x)|^{p+1}}{p+1} dx < 0,$$

the local solution u given by (1) in Theorem 1.1 blows up in finite time. Namely, there exists $0 < S_1 < \infty$ such that $\lim_{s \nearrow S_1} \|\nabla u(s,\cdot)\|_{L^2(\mathbb{R}^n)} = \infty$.

Let us consider the case $s = n/2$. We put

$$V(v) := \kappa \sum_{j \geq j_0} \frac{\alpha^j |v|^{\nu j+2}}{j!(\nu j+2)}, \tag{1.4}$$

$$V'(v) = \kappa \sum_{j \geq j_0} \frac{\alpha^j}{j!} |v|^{\nu j} v = \kappa \left\{ \exp(\alpha |v|^\nu) - \sum_{0 \leq j < j_0} \frac{\alpha^j}{j!} |v|^{\nu j} \right\} v,$$

where $\kappa \in \mathbb{C}$, $0 < \alpha < \infty$, $0 < \nu \leq 2$ and $j_0 \in \{1, 2, \cdots\}$. Then (1.1) is rewritten as

$$\begin{cases} i \dfrac{\partial u}{\partial s}(s,x) \pm \dfrac{1}{2} \Delta u(s,x) \mp \dfrac{\kappa}{2} \displaystyle\sum_{j \geq j_0} \dfrac{\alpha^j}{j!} b(s)^{n\nu j/4-1} |u(s,x)|^{\nu j} u(s,x) = 0, \\ u(0,\cdot) = u_0(\cdot) \in H^{n/2}(\mathbb{R}^n). \end{cases} \tag{1.5}$$

Theorem 1.4. *Let $n \geq 1$, $H \in \mathbb{R}$, $\kappa \in \mathbb{C}$. Let $\alpha > 0$, $0 < \nu \leq 2$. There exist two admissible pairs $\{(q_j, r_j)\}_{j=1,2}$ with the following properties.*

(1) (Local solutions.) For any initial data $u_0 \in H^{n/2}(\mathbb{R}^n)$, there exist $S > 0$ with $S \leq S_0$ and a unique time local solution u of (1.5) in $X^{n/2}([0,S))$, where we assume $\|u_0\|_{\dot{H}^{n/2}(\mathbb{R}^n)}$ is sufficiently small when $\nu = 2$.

(2) (Small global solutions.) Let $H \geq 0$ and $j_0 \geq 4/n\nu$. If $\|u_0\|_{L^2(\mathbb{R}^n)}$ is sufficiently small, then the solution u obtained in (1) is a global solution, namely, $S = S_0$. And u behaves as the free solution asymptotically.

For an admissible pair (q, r), we define the function space

$$Y([0,S)) := C([0,S), H^1(\mathbb{R}^2)) \cap L^\infty((0,S), H^1(\mathbb{R}^2)) \cap L^q((0,S), H^{1,r}(\mathbb{R}^2)).$$

Corollary 1.5. *If $\kappa > 0$, $H > 0$, and the energy of u_0 satisfies*

$$\int_{\mathbb{R}^2} |\nabla u_0(x)|^2 + \frac{\kappa}{\alpha} \left(e^{\alpha |u_0(x)|^2} - 1 - \alpha |u_0(x)|^2 \right) dx \leq \frac{4\pi}{\alpha}, \tag{1.6}$$

then the local solution u is a global solution.