Stability of non-isolated asymptotic profiles for fast diffusion

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1 Asymptotic profiles of vanishing solutions

Let Ω be a bounded domain of $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. We are concerned with the Cauchy-Dirichlet problem for Fast Diffusion Equation (FDE, for short),

$$\partial_t \left(|u|^{m-2}u\right) = \Delta u \quad \text{in } \Omega \times (0,\infty), \tag{1}$$

$$u = 0 \quad \text{on } \partial\Omega \times (0,\infty), \tag{2}$$

$$u(\cdot,0) = u_0 \quad \text{in } \Omega, \tag{3}$$

where $\partial_t = \partial/\partial t$, under the assumptions that

$$u_0 \in H^1_0(\Omega), \quad 2 < m < 2^* := \frac{2N}{(N-2)_+}. \tag{4}$$

By putting $w = |u|^{m-2}u$, Equation (1) is rewritten as a more usual form,

$$\partial_t w = \Delta \left(|w|^{r-2}w\right) \qquad \text{in } \Omega \times (0,\infty)$$

with the exponent $r = m' := m/(m-1) \in (1,2)$. In particular, FDE arises in Plasma Physics to describe anomalous diffusion of plasma in toroidal flow (see [4, 5, 6] and [32]).

Notation. We write $\|\cdot\|_{H^1_0(\Omega)} := \|\nabla\cdot\|_{L^2(\Omega)}$. For a function $u = u(x,t)$ from $\Omega \times (0,\infty)$ to $\mathbb{R}$, we often write $u(t) := u(\cdot,t)$, which is a function from Ω to $\mathbb{R}$, for each fixed time $t > 0$.

One of typical features of solutions to (FD) (:= (1)–(3)) is the *extinction in finite time*, namely, every solution vanishes at a finite time (see [34, 7, 18, 27]). Moreover, Berryman and Holland [5] determined the optimal extinction rate of solutions $u = u(x,t)$ vanishing at a finite time $t_* > 0$ under (4). More precisely, it holds that

$$c_1(t_* - t)_+^{1/(m-2)} \le \|u(t)\|_{H^1_0(\Omega)} \le c_2(t_* - t)_+^{1/(m-2)} \quad \text{for all } \ t \ge 0$$

with $c_1, c_2 > 0$, provided that $u_0 \not\equiv 0$. Here $t_* = t_*(u_0)$ is called *extinction time* (of the unique solution $u(x,t)$) for each data u_0. Then the *asymptotic profile* $\phi = \phi(x)$ of each solution $u = u(x,t)$ is defined by

$$\phi(x) := \lim_{t \nearrow t_*} (t_* - t)^{-1/(m-2)} u(x,t) \quad \text{in } H^1_0(\Omega).$$

In order to characterize ϕ, we apply the following transformation:

$$v(x,s) := (t_* - t)^{-1/(m-2)} u(x,t) \quad \text{and} \quad s := \log(t_*/(t_* - t)) \ge 0. \tag{5}$$

Then the asymptotic profile $\phi = \phi(x)$ of $u = u(x,t)$ is reformulated as

$$\phi(x) = \lim_{s \nearrow \infty} v(x,s) \quad \text{in } H^1_0(\Omega).$$

Moreover, (FD) is rewritten as the following rescaled problem (RP):

$$\partial_s \left(|v|^{m-2}v\right) = \Delta v + \lambda_m |v|^{m-2} v \quad \text{in } \ \Omega \times (0,\infty), \tag{6}$$

$$v = 0 \quad \text{on } \ \partial\Omega \times (0,\infty), \tag{7}$$

$$v(\cdot,0) = v_0 \quad \text{in } \ \Omega, \tag{8}$$

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where $v_0 = t_*(u_0)^{-1/(m-2)}u_0$ and $\lambda_m = (m-1)/(m-2) > 0$. This problem can be formulated as a (generalized) *gradient flow*,

$$\partial_s \left(|v|^{m-2}v\right)(s) = -J'(v(s)) \quad \text{in } H^{-1}(\Omega), \quad s > 0, \qquad v(0) = v_0,$$

where J' stands for the Fréchet derivative of the functional,

$$J(w) = \frac{1}{2}\|w\|^2_{H^1_0(\Omega)} - \frac{\lambda_m}{m}\|w\|^m_{L^m(\Omega)} \quad \text{for } w \in H^1_0(\Omega).$$

Hence $s \mapsto J(v(s))$ is nonincreasing. Then it follows that

Theorem 1 (Asymptotic profiles (Berryman and Holland [5]))

Let $u_0 \in H^1_0(\Omega) \setminus \{0\}$ and let v be a rescaled solution. For any sequence $s_n \to \infty$, there exist a subsequence (n') of (n) and $\phi \in H^1_0(\Omega) \setminus \{0\}$ such that $v(s_{n'}) \to \phi$ strongly in $H^1_0(\Omega)$. Moreover, ϕ solves the Emden-Fowler equation (EF):

$$-\Delta\phi = \lambda_m|\phi|^{m-2}\phi \quad \text{in} \quad \Omega, \quad \phi = 0 \quad \text{on} \quad \partial\Omega. \tag{9}$$

See also [29, 19, 35] and [8, 9, 10, 11, 12] for related results. In particular, the asymptotic profile ϕ is uniquely determined for each nonnegative data $u_0 \geq 0$ (see [20]).

One can also find that the set of all asymptotic profiles of solutions for (FD) coincides with the set of all nontrivial solutions of (EF) (= the set of all nontrivial critical points of J). Here and henceforth, we denote by $\mathcal{S}$ these sets.

2 Stability analysis of asymptotic profiles

In this talk, we address ourselves to the stability of asymptotic profiles. Namely, our question is whether or not solutions of (1)–(3) emanating from a small neighborhood of an asymptotic profile $\phi \in \mathcal{S}$ also have the same profile ϕ. In order to precisely formulate such a notion of stability, let us recall the transformation (5). Taking account of the relation, $v_0 = t_*(u_0)^{-1/(m-2)}u_0$, we need to introduce the phase set,

$$\mathcal{X} := \left\{t_*(u_0)^{-1/(m-2)}u_0 \colon u_0 \in H^1_0(\Omega) \setminus \{0\}\right\} = \left\{v_0 \in H^1_0(\Omega) \colon t_*(v_0) = 1\right\},$$

which is homeomorphic to a unit sphere in $H^1_0(\Omega)$ and includes all nontrivial solutions of (EF) (see [2, Propositions 6 and 10]). Then notions of (asymptotic) stability and instability of profiles are defined as follows:

Definition 2 (Stability and instability of profiles, Akagi-Kajikiya [2])

Let $\phi \in \mathcal{S}$.

(i) ϕ is said to be *stable*, if for any $\varepsilon > 0$ there exists $\delta > 0$ such that any solution v of (6), (7) satisfies

$$\sup_{s\in[0,\infty)} \|v(s) - \phi\|_{H^1_0(\Omega)} < \varepsilon,$$

whenever $v(0) \in \mathcal{X}$ and $\|v(0) - \phi\|_{H^1_0(\Omega)} < \delta$.

(ii) ϕ is said to be *unstable*, if ϕ is not stable.

(iii) ϕ is said to be *asymptotically stable*, if ϕ is stable, and moreover, there exists $\delta_0 > 0$ such that any solution v of (6), (7) satisfies

$$\lim_{s\nearrow\infty} \|v(s) - \phi\|_{H^1_0(\Omega)} = 0,$$

whenever $v(0) \in \mathcal{X}$ and $\|v(0) - \phi\|_{H^1_0(\Omega)} < \delta_0$.

Let d be the *least energy* of J over nontrivial solutions, i.e.,

$$d := \inf_{v \in \mathcal{S}} J(v), \quad \mathcal{S} = \{ \text{ nontrivial solutions of (EF) } \}.$$

A *least energy solution* ϕ of (EF) means $\phi \in \mathcal{S}$ satisfying $J(\phi) = d$. One can prove that every least energy solution of (EF) is sign-definite (i.e., positive or negative) by using strong maximum principle (see, e.g., [33] for variational analysis).

Theorem 3 (Stability criteria for isolated profiles [2])

Let ϕ be a least energy solution of (EF). Then the following (i) and (ii) hold:

(i) ϕ is a stable profile, if ϕ is isolated in $H_0^1(\Omega)$ from the other least energy solutions.

(ii) ϕ is an asymptotically stable profile, if ϕ is isolated in $H_0^1(\Omega)$ from the other sign-definite solutions. In particular, if ϕ is the unique positive solution of (EF), then ϕ is asymptotically stable in the sense of profile.

Let ϕ be a sign-changing solution of (EF). Then (iii) and (iv) below are satisfied.

(iii) ϕ is not an asymptotically stable profile.

(iv) ϕ is an unstable profile, if ϕ is isolated in $H_0^1(\Omega)$ from $\{\psi \in \mathcal{S} \colon J(\psi) < J(\phi)\}$.

In case $\Omega = B_N(R) := \{x \in \mathbb{R}^N \colon |x| < R\}$, it is well known by [21] that (EF) admits the unique positive radial solution ϕ and no other positive solution. Hence ϕ is the unique asymptotic profile of positive solutions for (FD). Moreover, by Theorem 3, the positive radial profile ϕ is asymptotically stable.

On the other hand, in case

$$\Omega = A_N(a, b) := \{x \in \mathbb{R}^N \colon a < |x| < b\}, \quad 0 < a < b,$$

Coffman [17] proved that the least energy solution is not radially symmetric, provided that $(b - a)/a \ll 1$ (see also [30, 13]). Therefore by rotational transform, least energy solutions form a continuum in $H_0^1(\Omega)$, and hence, they are beyond the scope of the stability criteria mentioned above.

Remark 4 (Instability of the positive radial profile in thin annular domains). It is also proved by Ni [31] that (EF) admits the unique positive radial solution. For thin annuli, $(b-a)/a \ll 1$, the radial positive solution does not attain the least energy, and therefore, it is also beyond the scope of Theorem 3. In [1], the positive radial profile turns out to be not asymptotically stable under some quantitative condition on the (relative) thickness of the annulus, and furthermore, it is unstable for $N = 2$ (see also [3]).

3 Stability of least energy profiles

The main purpose of this talk is to prove the stability of all (possibly non-isolated) least energy solutions for smooth bounded domains. To this end, we restrict ourselves to nonnegative solutions for (1)–(3) (and also those for (6)–(8)). Define a subset of $\mathcal{X}$ by

$$\mathcal{X}_+ := \{v_0 \in \mathcal{X} \colon v_0 \geq 0 \text{ a.e. in } \Omega\}.$$

Then one can see that the solution $v(\cdot, s)$ of (6)–(8) is lying on $\mathcal{X}_+$ for any $s > 0$, provided that the initial data v_0 belongs to $\mathcal{X}_+$ (indeed, the nonnegativity of $v(\cdot, s)$ is inherited from v_0). Moreover one can rewrite Definition 2 by replacing $\mathcal{X}$ with $\mathcal{X}_+$ to consider only nonnegative solutions of (6)–(8).

Our main result reads,

Theorem 5 (Stability of least energy profiles for FDE)

Let $\phi > 0$ be a least energy solution of (EF). Then ϕ is stable (in the sense of asymptotic profiles for (1)–(3)) under the flow on $\mathcal{X}_+$ generated by nonnegative solutions for (6)–(8).

Our proof of the theorem above will rely on a uniform extinction estimate of solutions for (1)–(3) (see [19]) as well as the so-called *Lojasiewicz-Simon inequality* (see [20]). Both devices are established for nonnegative solutions.

The Łojasiewicz-Simon inequality has been vigorously studied so far and usually employed to prove the convergence of each solution for nonlinear parabolic (and also damped wave) equations to a prescribed (possibly non-isolated) stationary solution as $t \to \infty$ (and hence, the ω-limit set of each evolutionary solution turns out to be singleton). More precisely, let $E : X \to \mathbb{R}$ be a "smooth" functional defined on a Banach space X and let ψ be a critical point of E (or a stationary point), i.e., $E'(\psi) = 0$ in the dual space X^*, where $E' : X \to X^*$ denotes the Fréchet derivative of E. Then an abstract form of the Łojasiewicz-Simon inequality is as follows (see, e.g., [36, 28, 25, 22, 20, 24, 26, 15, 16, 14, 23]): there exist constants $\theta \in (0, 1/2]$ and $\omega, \delta > 0$ such that

$$|E(v) - E(\psi)|^{1-\theta} \leq \omega \|E'(v)\|_{X^*} \quad \text{for all } v \in X \ \text{ satisfying } \ \|v - \psi\|_X < \delta$$

(cf. there are several variants with different choices of norms).

We close this section with precise statements of the uniform extinction estimate and the Łojasiewicz-Simon inequality, which will be used to prove Theorem 5.

Lemma 6 (Uniform extinction estimate for FDE [19])

Let $u_0 \in H^1_0(\Omega)$, $u_0 \geq 0$ and let $u = u(x,t)$ be the nonnegative solution of (1)–(3) with the initial data u_0. Then for each $t_0 \in (0, t_*/2]$, it holds that

$$\|u^{m-1}(t)\|_{L^\infty(\Omega)} \leq K\,(t_* - t)_+^{(m-1)/(m-2)} \qquad \text{for all } \ t \geq t_0,$$

where t_* is the extinction time of $u(x,t)$. Here K is a constant given by

$$K := \gamma \left(\frac{t_0}{t_* - t_0} \right)^{-\frac{N}{\kappa}} R(u_0)^{\frac{2N}{\kappa} + \frac{2(m-1)}{m-2}}, \quad \kappa := \frac{2N - Nm + 2m}{m-1} > 0$$

with a constant $\gamma = \gamma(N, m, |\Omega|) > 0$ and the Rayleigh quotient,

$$R(w) := \frac{\|w\|_{H^1_0(\Omega)}}{\|w\|_{L^m(\Omega)}} \quad \text{for } \ w \in H^1_0(\Omega).$$

Let ϕ be an arbitrary least energy solution of (EF). Since least energy solutions are sign-definite, we also assume $\phi \geq 0$ without any loss of generality. Moreover, by strong maximum principle, one can assure that

$$0 < \phi(x) < L_\phi := \|\phi\|_{L^\infty(\Omega)} + 1 \ \text{ for all } \ x \in \Omega \quad \text{ and } \quad \partial_\nu \phi < 0 \ \text{ on } \partial\Omega.$$

Then the following Łojasiewicz-Simon inequality holds:

Lemma 7 (Łojasiewicz-Simon inequality [20])

For any $L > L_\phi$, there exist constants $\theta \in (0, 1/2)$, $\omega, \delta_0 > 0$ such that

$$|J(w) - J(\phi)|^{1-\theta} \leq \omega \left\| J'(w) \right\|_{H^{-1}(\Omega)} \tag{10}$$

for all $w \in H_0^1(\Omega)$ satisfying $0 \leq w(x) \leq L$ for a.e. $x \in \Omega$ and $\|w - \phi\|_{H_0^1(\Omega)} < \delta_0$.

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