Stationary scattering theory on manifold with ends: Rellich's theorem, limiting absorption principle and radiation condition

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1 Setting and results

We present without proofs a part of our results in an ongoing joint project [IS2] with E. Skibsted (Aarhus University). In particular, here, Rellich's theorem, the limiting absorption principle and the radiation condition are discussed for the Schrödinger operator

$$H = H_0 + V; \quad H_0 = -\frac{1}{2}\Delta = \frac{1}{2}p_i^* g^{ij} p_j,\ p_i = -\mathrm{i}\partial_i,$$

on a connected Riemannian manifold (M, g) with asymptotically Euclidean and/or hyperbolic ends. Here Δ is the Laplace–Beltrami operator, and V is a potential-type perturbation.

We first introduce a set of assumptions on the geometry of (M, g) and on the perturbation V, and we state our main results. Our conditions are formulated abstractly in terms of a certain "escape function". They apply to a wide class of manifolds and potentials. See Section 2 for concrete examples satisfying the conditions.

Condition 1.1. Let (M, g) be a connected Riemannian manifold of dimension $d \geq 1$. There exist a function $r \in C^\infty(M)$ with image $r(M) = [1, \infty)$ and constants $c > 0$ and $r_0 \geq 2$ such that:

1. The gradient vector field $\nabla r \in \mathfrak{X}(M)$ is forward complete, i.e., the integral curve of ∇r exists for any initial point $x \in M$ and any non-negative time $t \geq 0$.

2. The bound $|\nabla r| \geq c$ holds on $\{x \in M;\ r(x) > r_0/2\}$.

We call each component of the open subset $E = \{x \in M;\ r(x) > r_0\}$ an *end* of M. The function r may model a distance function there. In fact, only with Condition 1.1 we can canonically construct the *spherical coordinates* on E as follows. Note that E is the union of r-spheres

$$S_R = \{x \in M;\ r(x) = R\}; \quad R > r_0.$$

By Condition 1.1 and the implicit function theorem these r-spheres are submanifolds of M, and we shall make use of the gradient vector field ∇r to canonically connect their coordinates. Choose $\chi \in C^\infty(\mathbb{R})$ such that

$$\chi(t) = \begin{cases} 1 & \text{for } t \le 1, \\ 0 & \text{for } t \ge 2, \end{cases} \qquad \chi \ge 0, \quad \chi' \le 0, \tag{1.1}$$

and define the *normalized* gradient vector field $X \in \mathfrak{X}(M)$ by

$$X = \eta |\nabla r|^{-2} \nabla r; \quad \eta = 1 - \chi(2r/r_0). \tag{1.2}$$

We let

$$y \colon \mathcal{M} \to M,\ (t,x) \mapsto y(t,x) = \exp(tX)(x); \quad \mathcal{M} \subseteq \mathbb{R} \times M, \tag{1.3}$$

be the maximal flow generated by the vector field X. By Condition 1.1 X is also forward complete, and $\mathcal{M}$ contains a neighborhood of $[0,\infty) \times M$ in $\mathbb{R} \times M$. By definition it satisfies, in local coordinates,

$$\partial_t y^i(t,x) = X^i(y(t,x)) = (\eta |\nabla r|^{-2} (\nabla r)^i)(y(t,x)), \quad y(0,x) = x.$$

This, in particular, implies that for any $x \in E$ and $t \ge 0$

$$r(y(t,x)) = r(x) + t,$$

and hence the semigroup (1.3) induces a family of diffeomorphic embeddings

$$\iota_{R,R'} = y(R' - R, \,\cdot\,)_{|S_R} \colon S_R \to S_{R'}; \quad R \le R' \tag{1.4}$$

satisfying

$$\iota_{R',R''} \circ \iota_{R,R'} = \iota_{R,R''}; \quad R \le R' \le R''. \tag{1.5}$$

Using (1.4) and (1.5) we may regard $S_R \subseteq S_{R'}$ for any $R \le R'$ in a well-defined manner, and this naturally induces a manifold structure on the union

$$S = \bigcup_{R > r_0} S_R. \tag{1.6}$$

In fact, such a manifold S is attained as an inductive limit (not to be elaborated on here). The manifold S may be considered as a boundary of M at infinity. We can define $\sigma(x) \in S$ for $x \in E$ by considering $x \in S_r \subseteq S$ with $r = r(x)$ and whence indeed identifying x with some $\sigma(x) \in S$. Let σ be any local coordinates on a neighbourhood of $\sigma(x) \in S$. Then the spherical coordinates of a point $x \in E$ are the components of (r, σ). Slightly inconsistently we may write $(r,\sigma) = (r(x), \sigma(x))$. We shall refer to r as the *radius function*, and $S_R \subseteq S$ as a *spherical manifold*. Note that in such coordinates the set E is identified with an open subset of the half-infinite cylinder $(r_0, \infty) \times S$ whose r-sections are monotonically increasing and exhausting S.

Condition 1.1 is a topological condition guaranteeing the existence of ends and the spherical coordinates there, and we need a more restrictive assumption that controls the geometry of E. Using η of (1.2), we introduce the tensor ℓ and the associated differential operator L by

$$\ell = g - \eta |\nabla r|^{-2} \mathrm{d}r \otimes \mathrm{d}r, \quad L = p_i^* \ell^{ij} p_j.$$

As we can see easily in the spherical coordinates, the tensor ℓ may be identified with the pull-back of g to the r-spheres S_R for $R > r_0$, and L with the spherical part of $-\Delta$. We remark that the tensor ℓ clearly satisfies

$$0 \le \ell \le g \text{ on } M, \quad \ell^{\bullet i}(\nabla r)_i = 0 \text{ on } E, \tag{1.7}$$

where the first bounds of (1.7) are understood as quadratic form estimates on fibers of the tangent bundle TM. Let us also recall a local expression of the Levi–Civita connection ∇: If we denote the Christoffel symbols by

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl}(\partial_i g_{lj} + \partial_j g_{li} - \partial_l g_{ij}),$$

then for any smooth function f on M

$$(\nabla f)_i = \nabla_i f = \partial_i f, \quad (\nabla^2 f)_{ij} = \partial_i \partial_j f - \Gamma^k_{ij} \partial_k f.$$

Note that $\nabla^2 f = \nabla\nabla f$ is the geometric Hessian of f.

Condition 1.2. There exist constants $\sigma, \tau, C > 0$ such that globally on M

$$r \nabla^2 r \ge \frac{1}{2} \sigma |\nabla r|^2 \ell - C r^{-\tau} g, \tag{1.8}$$

and for $\alpha = 0, 1$

$$\bigl|\nabla^\alpha |\nabla r|^2\bigr| \le C r^{-\alpha(1+\tau)}, \quad |\nabla^\alpha \Delta r| \le C, \quad |L\Delta r| \le C r^{-1-\tau}. \tag{1.9}$$

Condition 1.2 says that the ends are geometrically growing, which we remark is a different property from the monotonicity of the r-spheres mentioned above. For any $R > r_0$ we let $\iota_R \colon S_R \hookrightarrow M$ be the inclusion, and note the general identity

$$(\nabla^2 r)^{ij} (\nabla r)_j = \frac{1}{2} (\nabla |\nabla r|^2)^i. \tag{1.10}$$

Then it follows that, in case where r is an exact distance function, i.e., $|\nabla r| = 1$ on E, the Hessian $\nabla^2 r$ has no radial components in the spherical coordinates. Then $(\nabla^2 r)_{|S_R}$ is identified with the second fundamental form $\iota_R^*(\nabla^2 r)$ of S_R, and $(\Delta r)_{|S_R}$ is the mean curvature $\mathrm{tr}[\iota_R^*(\nabla^2 r)]$ of S_R. Here Under Condition 1.2, in general, the radial components of $\nabla^2 r$ does not vanish but are very small, and hence still we may regard $\nabla^2 r$ as the second fundamental form with negligible error. The bound (1.8) implies that the ends are growing, since it bounds the minimal curvature of S_R from below. The bounds in (1.9) are connected to the regularity properties of the mean curvature of S_R, and in particular, we can bound the maximal curvature from above, since $\iota_R^*(\nabla^2 r)$ is strictly positive for $R > r_0$ large enough. One benefit of our somewhat indirect description of the geometry of (M, g) is that it is obviously stable under small perturbations. Even when it is difficult to compute an exact distance function we may choose a more useful distance-like function to verify the assumptions.

Finally we would like to impose a long-range type condition on the potential V. More precisely, taking into account a metric quantity related to the volume growth of the ends, we formulate it in terms of the effective *joint potential* given as follows: Define

$$q = V + \frac{1}{8}\eta|\nabla r|^{-2}\big[(\Delta r)^2 + 2\nabla^r\Delta r\big]; \quad \nabla^r = (\nabla r)^i\nabla_i. \tag{1.11}$$

Condition 1.3. There exists a splitting by real-valued functions:

$$q = q_1 + q_2; \quad q_1 \in C^1(M) \cap L^\infty(M),\ q_2 \in L^\infty(M),$$

such that for some $\rho, C > 0$ the following bounds hold globally on M:

$$|\nabla^r q_1| \le Cr^{-1-\rho}, \quad |q_2| \le Cr^{-1-\rho}. \tag{1.12}$$

Now let us mention the self-adjoint realizations of H and H_0 on the Hilbert space $\mathcal{H} = L^2(M)$. Since (M, g) can be incomplete the operators H and H_0 are not necessarily essentially self-adjoint on $C_c^\infty(M)$. We realize H_0 as a self-adjoint operator by imposing the Dirichlet boundary condition, i.e. H_0 is the unique self-adjoint operator associated with the closure of the quadratic form

$$\langle H_0\rangle_\psi = -\frac{1}{2}\langle\psi, \Delta\psi\rangle = \frac{1}{2}\langle p\psi, p\psi\rangle, \quad \psi \in C_c^\infty(M).$$

We denote the form closure and the self-adjoint realization by the same symbol H_0. Define the associated Sobolev spaces $\mathcal{H}^s$ by

$$\mathcal{H}^s = (H_0 + 1)^{-s/2}\mathcal{H}, \quad s \in \mathbb{R}. \tag{1.13}$$

Then H_0 may be understood as a closed quadratic form on $Q(H_0) = \mathcal{H}^1$. Equivalently, H_0 makes sense also as a bounded operator $\mathcal{H}^1 \to \mathcal{H}^{-1}$ whose action coincides with that for distributions. By the definition of the Friedrichs extension the self-adjoint realization of H_0 is the restriction of such distributional $H_0\colon \mathcal{H}^1 \to \mathcal{H}^{-1}$ to the domain:

$$\mathcal{D}(H_0) = \{\psi \in \mathcal{H}^1;\ H_0\psi \in \mathcal{H}\} \subseteq \mathcal{H}.$$

Since V is bounded and self-adjoint by Conditions 1.1–1.3 we can realize the self-adjoint operator $H = H_0 + V$ simply as

$$H = H_0 + V, \quad \mathcal{D}(H) = \mathcal{D}(H_0).$$

In contrast to (1.13) we introduce the Hilbert spaces $\mathcal{H}_s$ with configuration weights:

$$\mathcal{H}_s = r^{-s}\mathcal{H}, \quad s \in \mathbb{R}.$$

We consider the r-balls $B_R = \{r(x) < R\}$ and the characteristic functions

$$F_\nu = F(B_{R_{\nu+1}} \setminus B_{R_\nu}),\ R_\nu = 2^\nu,\ \nu \geq 0, \tag{1.14}$$

where $F(\Omega)$ is used for sharp characteristic function of a subset $\Omega \subseteq M$. Define the associated Besov spaces B and B^* by

$$\begin{aligned} B &= \{\psi \in L^2_{\mathrm{loc}}(M);\ \|\psi\|_B < \infty\}, \quad \|\psi\|_B = \sum_{\nu=0}^{\infty} R_\nu^{1/2} \|F_\nu \psi\|_{\mathcal{H}}, \\ B^* &= \{\psi \in L^2_{\mathrm{loc}}(M);\ \|\psi\|_{B^*} < \infty\}, \quad \|\psi\|_{B^*} = \sup_{\nu \geq 0} R_\nu^{-1/2} \|F_\nu \psi\|_{\mathcal{H}}, \end{aligned} \tag{1.15}$$

respectively. We also define B^*_0 to be the closure of $C^\infty_{\mathrm{c}}(M)$ in B^*. Recall the nesting holding for any $s > 1/2$:

$$\mathcal{H}_s \subsetneq B \subsetneq \mathcal{H}_{1/2} \subsetneq \mathcal{H} \subsetneq \mathcal{H}_{-1/2} \subsetneq B^*_0 \subsetneq B^* \subsetneq \mathcal{H}_{-s}.$$

Using the function $\chi \in C^\infty(\mathbb{R})$ of (1.1), define $\chi_n, \bar{\chi}_n, \chi_{m,n} \in C^\infty(M)$ for $n > m \geq 0$ by

$$\chi_n = \chi(r/R_n), \quad \bar{\chi}_n = 1 - \chi_n, \quad \chi_{m,n} = \bar{\chi}_m \chi_n.$$

Let us introduce an auxiliary space:

$$\mathcal{N} = \{\psi \in L^2_{\mathrm{loc}}(M);\ \chi_n \psi \in \mathcal{H}^1 \text{ for all } n \geq 0\}.$$

This is the space of functions that satisfy the Dirichlet boundary condition, possibly with infinite $\mathcal{H}^1$-norm on M. Note that under Conditions 1.1–1.3 the manifold M may be, e.g., a half-space in the Euclidean space, and there could be a “boundary” even for large r, which is “invisible” from inside M.

Our first theorem is Rellich's theorem, the absence of B^*_0-eigenfunctions with eigenvalues above a certain *critical energy* $\lambda_H \in \mathbb{R}$. We set

$$\lambda_H = \limsup_{r \to \infty} q_1 = \lim_{R \to \infty} \bigl(\sup\{q_1(x);\ r(x) \geq R\}\bigr), \quad \mathcal{I} = (\lambda_H, \infty). \tag{1.16}$$

For many examples, including the Euclidean and the hyperbolic spaces, the essential spectrum is given by $\sigma_{\mathrm{ess}}(H) = [\lambda_H, \infty)$. This is seen in terms of Weyl sequences, see [K].

Theorem 1.4. *Suppose Conditions 1.1–1.3. If a function $\phi \in L^2_{\mathrm{loc}}(M)$ satisfies that*

1. *$\bar{\chi}_n \phi \in \mathcal{N} \cap B^*_0$ for some $n \geq 0$,*
2. *$(H - \lambda)\phi = 0$ for some $\lambda \in \mathcal{I}$ in the distributional sense,*

then $\phi = 0$ in M.

Corollary 1.5. *The operator H has no eigenvalues above λ_H: $\sigma_{\mathrm{pp}}(H) \cap \mathcal{I} = \emptyset$.*

In the statement of Theorem 1.4 we can drop the space $\mathcal{N}$ if the r-annuli $B_{R_{\nu+1}} \setminus B_{R_\nu}$ are relatively compact in M for all large $\nu \geq 0$. Note that an r-ball B_R, $R \geq 1$, may be unbounded under Conditions 1.1–1.3. Corollary 1.5 was proved in a somewhat similar setting in [IS1].

Next we discuss the limiting absorption principle and the radiation condition related to the resolvent

$$R(z) = (H - z)^{-1}, \quad z \in \mathbb{C} \setminus \sigma(H).$$

We first establish a locally uniform bound for the resolvent $R(z)$ as a map $B \to B^*$ under the following compactness condition.

Condition 1.6. In addition to Conditions 1.1–1.3 the embedding $r^{-s}\mathcal{H}^1 \hookrightarrow \mathcal{H}$ is compact for any $s > 0$.

In practice, due to Rellich's compact embedding theorem, a sufficient condition for Condition 1.6 is that M is complete and the r-balls B_R, $R \geq 1$, are bounded. More generally, even if M is incomplete, it suffices that M is isometrically embedded into some complete manifold, and the image of each B_R, $R \geq 1$, under this embedding is relatively compact.

For notational simplicity we fix large $C > 0$ and set

$$h = \nabla^2 r + 2Cr^{-1-\tau} g \geq Cr^{-1-\tau} g > 0,$$

cf. (1.7) and (1.8). By (1.10) the radial components of h are small, and we may consider h as the second fundamental form with negligible error. For any open subset $I \subseteq \mathcal{I}$ we denote

$$I_\pm = \{z = \lambda \pm \mathrm{i}\Gamma \in \mathbb{C} |\ \lambda \in I,\ \Gamma \in (0,1)\},$$

respectively. We also use the notation $\langle T \rangle_\phi = \langle \phi, T\phi \rangle$ and $p^r = -\mathrm{i}\nabla^r$.

Theorem 1.7. *Suppose Condition 1.6 and let $I \subseteq \mathcal{I}$ be any relatively compact open subset. Then there exists $C > 0$ such that for any $\phi = R(z)\psi$ with $z \in I_\pm$ and $\psi \in B$*

$$\|\phi\|_{B^*} + \|p^r \phi\|_{B^*} + \langle p_i^* h^{ij} p_j \rangle_\phi^{1/2} + \|H_0 \phi\|_{B^*} \leq C\|\psi\|_B. \tag{1.17}$$

In our theory the Besov boundedness (1.17) does not immediately imply the limiting absorption principle, and for that we need to establish radiation condition bounds. Let us impose an additional regularity assumption on the joint potential q.

Condition 1.8. In addition to requiring that Condition 1.6 holds, there exist splittings $q_1 = q_{11} + q_{12}$ and $q_2 = q_{21} + q_{22}$ by real-valued functions

$$q_{11} \in C^2(M) \cap L^\infty(M), \quad q_{12}, q_{21} \in C^1(M) \cap L^\infty(M), \quad q_{22} \in L^\infty(M)$$

and constants $\tilde{\rho}, C > 0$, such that for $\alpha = 0, 1$

$$|\nabla^r q_{11}| \leq Cr^{-(1+\tilde{\rho}/2)/2}, \qquad |\ell^{\bullet i}(\nabla q_{11})_i| \leq Cr^{-1-\tilde{\rho}/2}, \qquad |\nabla\nabla^r q_{11}| \leq Cr^{-1-\tilde{\rho}/2},$$
$$|\nabla q_{12}| \leq Cr^{-1-\tilde{\rho}/2}, \qquad |(\nabla^r)^\alpha q_{21}| \leq Cr^{-\alpha-\tilde{\rho}}, \qquad |q_{22}| \leq Cr^{-1-\tilde{\rho}/2}.$$

Our radiation condition bounds are stated in terms of a radial derivative A and an asymptotic complex phase a given below. Let A be a distributional differential operator expressed by

$$A = \mathrm{Re}\, p^r = \frac{1}{2}\big(p^r + (p^r)^*\big); \quad p^r = -\mathrm{i}\nabla^r = -\mathrm{i}(\nabla r)^i \nabla_i.$$

Pick a smooth non-increasing function $r_\lambda \geq r_0$ of $\lambda > \lambda_H$ such that

$$\lambda + \lambda_H - 2q_1 \geq 0 \quad \text{for } r \geq \frac{1}{2} r_\lambda,$$

and that $r_\lambda = r_0$ for all λ large enough. Then we set for $z = \lambda \pm \mathrm{i}\Gamma \in \mathcal{I} \cup \mathcal{I}_\pm$

$$a = a_z = \eta_\lambda \left(|\nabla r| \sqrt{2(z - q_1)} \pm \frac{p^r q_{11}}{4(z - q_1)} \right); \quad \eta_\lambda = 1 - \chi(2r/r_\lambda), \tag{1.18}$$

respectively, where the branch of square root is chosen such that $\mathrm{Re}\sqrt{w} > 0$ for $w \in \mathbb{C} \setminus (-\infty, 0]$. Note that the phase a of (1.18) is an approximate solution to the radial Riccati equation

$$\pm p^r a + a^2 - 2|\nabla r|^2 (z - q_1) = 0 \tag{1.19}$$

in the sense that it makes the quantity on the left-hand side of (1.19) small for large $r \geq 1$. The first term in the brackets of (1.18) alone already gives an approximate solution to the same equation, however, with the second term a better approximation is obtained. Set

$$\tilde{\beta} = \frac{1}{2} \min\{\tilde{\rho}, \sigma, \tau\} > 0. \tag{1.20}$$

Theorem 1.9. *Suppose Condition 1.8, and let $I \subseteq \mathcal{I}$ be any relatively compact open subset. Then for all $\beta \in [0, \tilde{\beta})$ there exists $C > 0$ such that for any $\phi = R(z)\psi$ with $\psi \in r^{-\beta} B$ and $z \in I_\pm$*

$$\|r^\beta (A \mp a)\phi\|_{B^*} + \langle p_i^* r^{2\beta} h^{ij} p_j \rangle_\phi^{1/2} \leq C \|r^\beta \psi\|_B,$$

respectively.

As an application we obtain the limiting absorption principle.

Corollary 1.10. *Suppose Condition 1.8, and let $I \subseteq \mathcal{I}$ be any relatively compact open subset. For any $s > 1/2$ and $\epsilon \in (0, \min\{(2s-1)/(2s+1), \tilde{\beta}/(\tilde{\beta}+1)\})$ there exists $C > 0$ such that for all $z, z' \in I_+$ or $z, z' \in I_-$ and $\alpha = 0, 1$*

$$\|p^\alpha R(z) - p^\alpha R(z')\|_{\mathcal{B}(\mathcal{H}_s, \mathcal{H}_{-s})} \leq C |z - z'|^\epsilon.$$

In particular, the operator $p^\alpha R(z)$ attains uniform limits as $I_\pm \ni z \to \lambda \in I$ in the norm topology of $\mathcal{B}(\mathcal{H}_s, \mathcal{H}_{-s})$:

$$p^\alpha R(\lambda \pm \mathrm{i}0) := \lim_{I_\pm \ni z \to \lambda} p^\alpha R(z), \quad \lambda \in I,$$

respectively. These limits $R(\lambda \pm \mathrm{i}0) \in \mathcal{B}(B, B^*)$ and map into $\mathcal{N}$.

Corollary 1.11. *The operator H has no singular continuous spectrum above λ_H: $\sigma_{\mathrm{sc}}(H) \cap \mathcal{I} = \emptyset$.*

Absence of singular continuous spectrum is a standard application of the limiting absorption principle.

Now we have the limiting resolvents $R(\lambda \pm \mathrm{i}0)$, and the radiation condition bounds for real spectral parameters follow directly from Theorem 1.9.

Corollary 1.12. *Suppose Condition 1.8, and let $I \subseteq \mathcal{I}$ be any relatively compact open subset. Then for all $\beta \in [0, \tilde{\beta})$ there exists $C > 0$ such that for any $\phi = R(\lambda \pm \mathrm{i}0)\psi$ with $\psi \in r^{-\beta}B$ and $\lambda \in I$*

$$\|r^\beta (A \mp a)\phi\|_{B^*} + \langle p_i^* r^{2\beta} h^{ij} p_j \rangle_\phi^{1/2} \leq C \|r^\beta \psi\|_B, \tag{1.21}$$

respectively.

As another application of the radiation condition bounds we can characterize the limiting resolvents $R(\lambda \pm \mathrm{i}0)$. For the Euclidean space such characterization is usually referred to as the *Sommerfeld uniqueness result.*

Corollary 1.13. *Suppose Condition 1.8, and let $\lambda \in \mathcal{I}$, $\phi \in L^2_{\mathrm{loc}}(M)$ and $\psi \in r^{-\beta}B$ with $\beta \in [0, \tilde{\beta})$. Then $\phi = R(\lambda \pm \mathrm{i}0)\psi$ holds if and only if both of the following conditions hold:*

1. *$\phi \in \mathcal{N} \cap r^\beta B^*$ and $(A \mp a)\phi \in r^{-\beta} B_0^*$.*
2. *$(H - \lambda)\phi = \psi$ in the distributional sense.*

2 Examples

In this section we provide several examples of Riemannian manifolds (M, g) satisfying Condition 1.8. For simplicity we consider the case where $V \equiv 0$. We let (M, g) be a complete Riemannian manifold, and assume that M has explicit end structure as follows: Suppose that there exist a compact subset $K \subseteq M$ and a closed Riemannian manifold (S, h) of dimension $d - 1$ such that the closure

$$\overline{(M \setminus K)} \cong [2, \infty) \times S,$$

and that in the "spherical coordinates" $(r, \sigma) \in (2, \infty) \times S$ the metric g is of warped-product type:

$$g(r, \sigma) = \mathrm{d}r \otimes \mathrm{d}r + f(r) h(\sigma); \quad h(\sigma) = h_{\alpha\beta}(\sigma)\, \mathrm{d}\sigma^\alpha \otimes \mathrm{d}\sigma^\beta. \tag{2.1}$$

Here the Greek indices run over $2, \ldots, d$. We may extend r suitably to K to be a globally defined smooth function with image $r(M) = [1, \infty)$. Then, if we choose $r_0 = 4$, Condition 1.1 and the compactness condition stated in Condition 1.6 are clearly satisfied. Below we are goint to examine the other bounds in Condition 1.8 by specifying the growth rate f of the ends explicitly. Note that the Christoffel symbols are computed as

$$\Gamma^r_{rr} = \Gamma^r_{r\alpha} = \Gamma^r_{\alpha r} = \Gamma^\alpha_{rr} = 0, \quad \Gamma^r_{\alpha\beta} = -\frac{1}{2} f' h_{\alpha\beta}, \quad \Gamma^\alpha_{r\beta} = \Gamma^\alpha_{\beta r} = \frac{1}{2}\frac{f'}{f}\delta^\alpha{}_\beta,$$

where $\delta^\alpha{}_\beta$ denotes Kronecker's δ. Then, on the ends E

$$|\nabla r|^2 = 1, \quad \nabla^2 r = \frac{1}{2} f' h, \quad \Delta r = \frac{d-1}{2}\frac{f'}{f}, \quad L\Delta r = 0,$$

and, since $V \equiv 0$,

$$q = \frac{d-1}{8}\left[\frac{d-1}{4}\left(\frac{f'}{f}\right)^2 + \left(\frac{f'}{f}\right)'\right].$$

Now we can verify Condition 1.8 for the following examples.

Examples 2.1. 1. If

$$f(r) = r^\theta; \quad \theta > 0,$$

then Condition 1.8 is satisfied for $\sigma = \theta$, arbitrary $\tau > 0$, $\rho = 2$ and $\tilde{\rho} = 6$ with $q_1 = q_{11} = q$, and the critical energy is $\lambda_H = 0$. The Euclidean space corresponds to $f(r) = r^2$ and S being the standard unit sphere.

2. If

$$f(r) = \exp(\delta r^\theta); \quad \delta > 0, \ \theta \in (0,1),$$

then Condition 1.8 is satisfied for arbitrary $\sigma > 0$, $\tau > 0$, $\rho = 2-2\theta$ and $\tilde{\rho} = 6-4\theta$ with $q_1 = q_{11} = q$, and the critical energy is $\lambda_H = 0$.

3. Let

$$f(r) = \exp(\kappa r + \delta_\theta(r)),$$

where $\kappa > 0$ and $\theta < 1$ are constants and δ_θ is a function of r satisfying

$$|\delta_\theta^{(k)}(r)| \le C r^{\theta - k} \quad \text{for } k = 1,2,3,4.$$

Then Condition 1.8 is satisfied for arbitrary $\sigma > 0$, $\tau > 0$, $\rho = 1-\theta$ and $\tilde{\rho} = 4-2\theta$ with $q_1 = q_{11} = q$, and the critical energy is $\lambda_H = (d-1)^2\kappa^2/32$. The hyperbolic space corresponds to $f(r) = (\sinh r)^2$ and S being the standard unit sphere, for which $\theta < 1$ may be arbitrary.

We can further perturb the models of Examples 2.1. For example, we can add to (2.1) suitable lower order terms, whether warped-product type or not. We can also put any compact obstacle or attach topological handles to these manifolds. Obstacles can be also non-compact, as long as the gradient vector field ∇r is inward pointing on the boundary as consistent with the following more general example.

Example 2.2. Let (M, g) be any of the Riemannian manifolds discussed in Examples 2.1. Let $\Omega \subseteq M$ be a connected open subset such that the r-sections

$$S_{\Omega,R} = \Omega \cap (\{R\} \times S),$$

regarded as a subset of S, are monotonically increasing for large $R > r_0$. If we appropriately modify the function r for its small values, then Ω satisfies Condition 1.8 with the same parameters σ, τ, ρ and $\tilde{\rho}$ as those of M. This example in particular includes the solid cones and the half-spaces of the Euclidean and the hyperbolic spaces. More general regions such as the complement of a solid paraboloid in the Euclidean space are also included.

References

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