INTRINSIC ULTRACONTRACTIVITY VIA CAPACITARY WIDTH

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1. Introduction

Let D be a domain in $\mathbb{R}^n$, $n \ge 2$. Let $p_D(t,x,y)$, $t>0$, $x,y \in D$, be the heat kernel for $\Delta - \partial/\partial t$ on D, i.e., $u(t,x) = \int_D p_D(t,x,y) f(y) dy$ is the solution to

$$\begin{aligned} \Delta u - \frac{\partial u}{\partial t} &= 0 \quad \text{on } (0,\infty)\times D, \\ u &= 0 \quad \text{on } (0,\infty)\times \partial D, \\ u &= f \quad \text{on } \{0\}\times D. \end{aligned}$$

Definition 1.1 (Davies-Simon [DS84])**.** We say that D *intrinsically ultracontractive* (abbreviated to IU) if the following two conditions are satisfied:

(i) The eigenvalue problem $-\Delta u = \lambda u$ in D subject to the Dirichlet boundary condition $u = 0$ on ∂D has the first eigenvalue $\lambda_D > 0$ with corresponding positive eigenfunction φ_D normalized by $\|\varphi_D\|_2 = 1$.

(ii) For every $t > 0$, there exist constants $0 < c_t < 1 < C_t$ depending on t such that

$$c_t\varphi_D(x)\varphi_D(y) \le p_D(t,x,y) \le C_t\varphi_D(x)\varphi_D(y) \quad \text{for all } x,y \in D. \tag{1.1}$$

Remark 1.2. We can replace (i) by

(i') The Dirichlet Laplacian $-\Delta$ has no essential spectrum; and hence the heat kernel $p_D(t,x,y)$ has eigenfunction expansion

$$p_D(t,x,y) = \sum_{j=0}^{\infty} e^{-\lambda_j t}\varphi_j(x)\varphi_j(y),$$

where $\lambda_0 < \lambda_1 \le \lambda_2 \le \cdots$ and $\varphi_0, \varphi_1, \varphi_2, \ldots$ are eigenvalues and eigenfunctions. Here $\lambda_D = \lambda_0$ and $\varphi_D = \varphi_0$ for notational convenience.

See [Dav89] for further details.

We shall see that (i') is completely characterized by *capacitary width*. Roughly speaking, if D is *small* at infinity, then (i') holds. On the other hand, (ii) is a *smoothness* condition on D. It is global, very mild and subtle and has been fascinating a number of mathematicians. This is our main objective.

This introduction provides several known results to increase the reader's familiarity with IU. First, we observe that if the upper estimate of (1.1) holds at some time, say t_0, then so does it after t_0 with exponentially decaying constant C_t.

Proposition 1.3. *Suppose $p_D(t_0,x,y) \le C_{t_0}\varphi_D(x)\varphi_D(y)$ for all $x,y \in D$ with some $t_0 > 0$. If $t \ge t_0$, then*

$$p_D(t,x,y) \le C_{t_0}e^{-\lambda_D(t-t_0)}\varphi_D(x)\varphi_D(y) \quad \textit{for all } x,y \in D.$$

In other words, $p_D(t,x,y) \le C_t\varphi_D(x)\varphi_D(y)$ holds with $C_t \le C_{t_0}e^{-\lambda_D(t-t_0)}$ for $t \ge t_0$.

Proof. Fix $y \in D$. It is easy to see that the function $u(t,x) = C_{t_0} e^{-\lambda_D(t-t_0)}\varphi_D(x)\varphi_D(y)$ is a caloric function with vanishing lateral boundary values. By assumption $p_D(t_0,x,y) \le u(t_0,x)$ for $x \in D$. Hence the comparison principle on $(t_0,\infty)\times D$ yields the required inequality. □

This simple observation of IU readily yields the Cranston-McConnell inequality, or the lifetime estimate.

Proposition 1.4 (Cranston-McConnell inequality)**.** *If u is a nonnegative superharmonic function in D, then*

$$\text{(CM)} \qquad \frac{1}{u(x)}\int_D G(x,y)u(y)dy \le A.$$

In the probabilistic language, $E_x^u(\tau_D) \le A$, where τ_D is the first exit time of the Brownian motion and $E_x^u(\tau_D)$ is the Doob's u-conditioned expectation with respect to Brownian motion starting from x.

Proof. Observe $G(x,y) = \int_0^\infty p_D(t,x,y)dt$. Regarding $u(x)$ as a time-invariant caloric function, we obtain from the comparison principle that

$$\int_D p_D(t,x,y)u(y)dy \le u(x) \quad \text{for every } t > 0.$$

Let $t_0 > 0$. We have

$$\int_0^{t_0} dt \int_D p_D(t,x,y)u(y)dy \le \int_0^{t_0} u(x)dt = t_0 u(x), \tag{1.2}$$

$$u(x) \ge \int_D p_D(t_0,x,y)u(y)dy \ge \int_D c_{t_0}\varphi_D(x)\varphi_D(y)u(y)dy \tag{1.3}$$

by the lower estimate of (1.1). Hence Proposition 1.3 and (1.3) yield

$$\begin{aligned}\int_{t_0}^\infty dt\int_D p_D(t,x,y)u(y)dy &\le \int_{t_0}^\infty dt \int_D C_{t_0}e^{-\lambda_D(t-t_0)}\varphi_D(x)\varphi_D(y)u(y)dy\\ &\le \frac{C_{t_0}}{c_{t_0}}\int_{t_0}^\infty dt\int_D e^{-\lambda_D(t-t_0)}p_D(t_0,x,y)u(y)dy \le \frac{A}{\lambda_D}u(x).\end{aligned}$$

Adding this inequality and (1.2), we obtain the Cranston-McConnell inequality. □

Remark 1.5. Cranston-McConnell [CM83] proved (CM) for a planar domain of finite area with A being a multiple of the area of D. Their result is remarkable since no regularity of D is needed. In the higher dimensional case, there exists a bounded domain (and hence of finite volume) for which (CM) fails. It is also known that there exists a planar domain of infinite area for which (CM) holds.

The eigenfunction expansion of the heat kernel yields very precise estimates for large $t > 0$. In particular, we see that the upper estimate of (1.1) automatically implies the lower estimate. In the following lemma, we write λ_0 and φ_0 for λ_D and φ_D, respectively for the notational convenience.

Proposition 1.6. *Let $\lambda_0 < \lambda_1 \le \lambda_2 \le \cdots$ be eigenvalues for $-\Delta u = 0$ in D and $u = 0$ on ∂D and let $\varphi_0, \varphi_1, \varphi_2, \dots$ be corresponding eigenfunctions normalized $\|\varphi_j\|_2 = 1$. Moreover let $\varphi_0 > 0$ on D.*

(i) $p_D(t,x,y) = \sum_{j=0}^{\infty} e^{-\lambda_j t}\varphi_j(x)\varphi_j(y)$.

(ii) $\sum_{j=0}^{\infty} e^{-\lambda_j t} \le A_{t_0} e^{-\lambda_1(t-t_0)}$ *for* $t \ge t_0$ *with* $A_{t_0} = \int_D p_D(t_0,x,x)dx$.

Moreover, if $p_D(t_0,x,x) \le c_{t_0}\varphi_0(x)^2$ *for every* $x \in D$ *(in particular, if D is IU), then the following hold:*

(iii) $\left|\frac{\varphi_j(x)}{\varphi_0(x)}\right|^2 \le c_{t_0} e^{\lambda_j t_0}$ *for* $j = 1,2,\dots$.

(iv) $\left|\frac{e^{\lambda_0 t} p_D(t,x,y)}{\varphi_0(x)\varphi_0(y)} - 1\right| \le C_{t_0} e^{-(\lambda_1-\lambda_0)t}$ *for* $t \ge 3t_0$.

(v) *For each* $\varepsilon > 0$ *there exists* $t(\varepsilon) > 0$ *such that*

$$(1-\varepsilon)e^{-\lambda_0 t}\varphi_0(x)\varphi_0(y) \le p_D(t,x,y) \le (1+\varepsilon)e^{-\lambda_0 t}\varphi_0(x)\varphi_0(y) \quad \textit{for } t \ge t(\varepsilon).$$

Proof. We have (i) by an eigenfunction expansion. If $t \ge t_0$, then by (i)

$$A_{t_0} = \int p_D(t_0,x,x)dx = \int \sum_{j=0}^{\infty} e^{-\lambda_j t_0}\varphi_j(x)^2 dx = \sum_{j=0}^{\infty} e^{-\lambda_j t_0} \ge \sum_{j=1}^{\infty} e^{\lambda_j(t-t_0)-\lambda_j t} \ge e^{\lambda_1(t-t_0)} \sum_{j=1}^{\infty} e^{-\lambda_j t},$$

which shows (ii). Now suppose $p_D(t_0,x,x) \le c_{t_0}\varphi_0(x)^2$. Then, by (i),

$$c_{t_0}\varphi_0(x)^2 \ge p_D(t_0,x,x) = \sum_{j=0}^{\infty} e^{-\lambda_j t_0}\varphi_j(x)^2 \ge e^{-\lambda_j t_0}\varphi_j(x)^2,$$

which gives (iii). Observe from (i) that

$$\frac{e^{\lambda_0 t} p_D(t,x,y)}{\varphi_0(x)\varphi_0(y)} - 1 = \sum_{j=1}^{\infty} e^{(\lambda_0-\lambda_j)t} \frac{\varphi_j(x)\varphi_j(y)}{\varphi_0(x)\varphi_0(y)}.$$

If $t \ge t_0$, then, from (iii), the right hand side is bounded in modulus by

$$\sum_{j=1}^{\infty} e^{(\lambda_0-\lambda_j)t} c_{t_0} e^{\lambda_j t_0} = c_{t_0} e^{\lambda_0 t} \sum_{j=1}^{\infty} e^{-\lambda_j(t-t_0)/2} e^{-\lambda_j(t-t_0)/2} \le c_{t_0} e^{\lambda_0 t} e^{-\lambda_1(t-t_0)/2} \sum_{j=1}^{\infty} e^{-\lambda_j(t-t_0)/2}.$$

By (ii) the last summation is not greater than $A_{t_0} e^{-\lambda_1[(t-t_0)/2-t_0]}$ if $(t-t_0)/2 \ge t_0$, i.e., $t \ge 3t_0$. These observations altogether yield

$$\left|\frac{e^{\lambda_0 t} p_D(t,x,y)}{\varphi_0(x)\varphi_0(y)} - 1\right| \le c_{t_0} A_{t_0} e^{\lambda_0 t} e^{\lambda_1 t_0} e^{-\lambda_1(t-t_0)} = c_{t_0} A_{t_0} e^{2\lambda_1 t_0} e^{-(\lambda_1-\lambda_0)t},$$

which shows (iv). It follows from (iv) that $\frac{e^{\lambda_0 t} p_D(t,x,y)}{\varphi_0(x)\varphi_0(y)} = 1 + o(1)$ as $t \to \infty$, which readily implies (v). □

2. Main results

Let us now state our results. See [Aik] for details.

2.1. **General sufficiency with Green function.**

Definition 2.1. Let $0 < \eta < 1$. For an open set D we define the *capacitary width* $w_\eta(D)$ by

$$w_\eta(D) = \inf\Big\{r > 0 : \frac{\mathrm{Cap}_{B(x,2r)}(B(x,r) \setminus D)}{\mathrm{Cap}_{B(x,2r)}(B(x,r))} \ge \eta \quad \text{for all } x \in D\Big\}.$$

Theorem 2.2. *The first eigenvalue λ_D satisfies*

$$\frac{A^{-1}}{w_\eta(D)^2} \le \lambda_D \le \frac{A}{w_\eta(D)^2},$$

where $A > 1$ depends only on η and n. In particular, there is no essential spectrum of $-\Delta$ if and only if $\lim_{R\to\infty} w_\eta(D \setminus \overline{B}(0,R)) = 0$.

Remark 2.3. The first eigenvalue, or the principal frequency, is a fascinating subject drawing a lot of attention. For instance, the relationship between the principal frequency and the volume is known as the Faber-Krahn inequality. In the complex analysis, it is estimated by the *inradius* for a simply connected domain. Very precise estimates are known under geometrical assumptions on D. The above estimate holds for arbitrary domains. Surprisingly enough, such an estimate has not known until Maz'ya-Shubin [MS05] proved the essentially same inequality with a different quantity (the dual of $w_\eta(D)$). We prove the theorem by a rather easy parabolic argument, inspired by Souplet [Sou00].

We proved the scale-invariant boundary Harnack principle, or local boundary Harnack principle (LBHP) for a uniform domain in [Aik01]. The following estimate of harmonic measure played a crucial role.

Proposition 2.4. *By $\omega^x(E, D)$ we denote the harmonic measure of E in D, evaluated at x. Let D be an open set, $x \in D$ and $R > 0$. Then*

$$\omega^x(D \cap \partial B(x,R), D \cap B(x,R)) \le A_0 \exp\Big(-\frac{A_1 R}{w_\eta(D)}\Big),$$

where positive constants A_0 and A_1 depend only on η and n.

Proposition 2.4 has a parabolic counterpart.

Proposition 2.5. *Let $P(t,x,D) = \int_D p_D(t,x,y)dy$. There exist positive constants A_2 and A_3 depending only on η and n such that*

$$P(t,x,D) \le A_2 \exp\Big(-\frac{A_3 t}{w_\eta(D)^2}\Big) \quad \textit{for all } t > 0 \textit{ and } x \in D.$$

The above propositions show similarity between an elliptic problem and parabolic problem. With the aid of Proposition 2.4, we ([Aik01]) developed the box argument, which is a combination of careful decomposition of the domain and repeated application of the maximum principle. It was inspired by Bass-Burdzy [BB92]. With the aid of Proposition 2.5, we have a parabolic counterpart, which may be referred to as a *parabolic box argument*. We can give general sufficient conditions in terms of capacitary width for IU and the global boundary Harnack principle (GBHP) in a unified fashion. For $f(x) > 0$ on D and $t > 0$ we write $w_\eta(f < t) = w_\eta(\{x \in D : f(x) < t\})$.

Theorem 2.6. *Let $g = G(\cdot, x_0)$ with $x_0 \in D$.*

(i) *If* $\displaystyle\int_0^1 w_\eta(g<t)^2\frac{dt}{t}<\infty$, *then* D *is IU.*

(ii) *If* $\displaystyle\int_0^1 w_\eta(g<t)\frac{dt}{t}<\infty$, *then* D *satisfies the GBHP.*

2.2. **Geometrical sufficient conditions.** Theorems 2.6 yields many sufficient conditions for IU and the GBHP. Define *quasihyperbolic metric* $k_D(x,y)$ by

$$k_D(x,y)=\inf_\gamma\int_\gamma\frac{ds}{\delta_D(\gamma(s))}\quad\text{for } x,y\in D,$$

where the infimum is taken over all rectifiable curves γ connecting x and y in D; γ is parameterized as $\gamma(s)$, $0\le s\le\ell(\gamma)$, by arc length s with $\ell(\gamma)$ being the length of γ. If h is a positive harmonic function in D, then

$$\exp(-Ak_D(x,y))\le\frac{h(y)}{h(x)}\le\exp(Ak_D(x,y))\quad\text{for } x,y\in D,$$

where $A>0$ depends only on n. In particular,

$$\{x\in D:g(x)<t\}\subset\{x\in D:k_D(x,x_0)>A\log(1/t)\}.$$

The change of variable $s=A\log(1/t)$ gives

Corollary 2.7. *The following statements hold:*

(i) *If* $\displaystyle\int_0^\infty w_\eta(k_D(\cdot,x_0)>s)^2ds<\infty$, *then* D *is IU.*

(ii) *If* $\displaystyle\int_0^\infty w_\eta(k_D(\cdot,x_0)>s)ds<\infty$, *then* D *satisfies the GBHP.*

Let $\Phi(t)$ be a positive nondecreasing continuous function of $t>0$ with $\Phi(0)=0$ and let

$$L^\Phi(D)=\Big\{f:\int_D\Phi(|f(x)|)dx<\infty\Big\}.$$

If $\Phi(t)=t^p$, then $L^\Phi(D)=L^p(D)$.

Theorem 2.8. *The following statements hold.*

(i) *Let* $n=2$. *If* $\log_+(1/g)\in L^1(D)$, *then* D *is IU.*

(ii) *Let* $n\ge 3$. *Suppose*

$$\int_1^\infty\Big(\frac{t}{\Phi(t)}\Big)^{2/(n-2)}dt<\infty. \tag{2.1}$$

If $\log_+(1/g)\in L^\Phi(D)$, *then* D *is IU.*

(iii) *Let* $n\ge 2$. *Suppose*

$$\int_1^\infty\Big(\frac{t}{\Phi(t)}\Big)^{1/(n-1)}dt<\infty. \tag{2.2}$$

If $\log_+(1/g)\in L^\Phi(D)$, *then* D *satisfies the GBHP.*

Remark 2.9. (i) Let $n\ge 3$. Typical examples of $\Phi(t)$ satisfying (2.1) are $\Phi(t)=t^p$ with $p>n/2$ and $\Phi(t)=t^{n/2}\log^\alpha(e+t)$ with $\alpha>(n-2)/2$. See [Cip94, Theorem 3].

(ii) Let $n\ge 2$. Typical examples of $\Phi(t)$ satisfying (2.2) are $\Phi(t)=t^p$ with $p>n$ and $\Phi(t)=t^n\log^\alpha(e+t)$ with $\alpha>n-1$.

Corollary 2.10. *The following statements hold.*

(i) *Let* $n = 2$. *If* $k_D(\cdot, x_0) \in L^1(D)$*, then* D *is IU.*
(ii) *Let* $n \geq 3$ *and suppose* Φ *satisfies* (2.1)*. If* $k_D(\cdot, x_0) \in L^\Phi(D)$*, then* D *is IU (cf.* [Cip94, Theorem 6]*).*
(iii) *Let* $n \geq 2$ *and suppose* Φ *satisfies* (2.2)*. If* $k_D(\cdot, x_0) \in L^\Phi(D)$*, then* D *satisfies the GBHP.*

Remark 2.11. The sufficient conditions for the GBHP in Theorem 2.8 and Corollary 2.10 are new even if $\Phi(t) = t^p$ with $p > n$.

Let us consider families of domains defined by conditions in terms of the quasihyperbolic metric.

Definition 2.12. We say that D satisfies the quasihyperbolic boundary condition of order α (QHB (α)) if

$$k_D(x, x_0) \leq A\Big(\frac{\delta_D(x_0)}{\delta_D(x)}\Big)^\alpha + A' \quad \text{for all } x \in D$$

for $\alpha > 0$. We say that D satisfies the QHB (0) if

$$k_D(x, x_0) \leq A \log \frac{\delta_D(x_0)}{\delta_D(x)} + A' \quad \text{for all } x \in D$$

We say that D satisfies the QHB(Φ) condition if

$$k_D(x, x_0) \leq \Phi\Big(\frac{\delta_D(x_0)}{\delta_D(x)}\Big) \quad \text{for all } x \in D.$$

Definition 2.13. Let $0 < r_0 \leq \infty$. We say that D satisfies the capacity density condition (abbreviated to CDC) up to r_0 if there exists positive constant η such that

$$\frac{\mathrm{Cap}_{B(x,2r)}(B(x, r) \setminus D)}{\mathrm{Cap}_{B(x,2r)}(B(x, r))} \geq \eta$$

whenever $x \in \partial D$ and $0 < r < r_0$. We simply say that D satisfies the CDC if it satisfies the CDC up to r_0 for some $r_0 > 0$.

Theorem A ([Bañ91] and [Aik09])**.** *Suppose* D *satisfies the CDC.*

(i) *If* D *satisfies the QHB(*α*) with* $0 \leq \alpha < 2$*, then* D *is IU.*
(ii) *If* D *satisfies the QHB(*α*) with* $0 \leq \alpha < 1$*, then the GBHP holds.*

Theorem 2.14. *Suppose* D *satisfies the CDC and the QHB(*Φ*).*

(i) *If* $\displaystyle\int_1^\infty \Phi(t)\frac{dt}{t^3} < \infty$*, then* D *is IU.*
(ii) *If* $\displaystyle\int_1^\infty \Phi(t)\frac{dt}{t^2} < \infty$*, then* D *satisfies the GBHP.*

Remark 2.15. (i) Typical examples of $\Phi(t)$ satisfying (i) are $\Phi(t) = t^\alpha$ with $\alpha < 2$ and $\Phi(t) = t^2 \log^{-\alpha}(e + t)$ with $\alpha > 1$. See [Bañ91, Corollary 2.8].
(ii) Typical examples of $\Phi(t)$ satisfying (ii) are $\Phi(t) = t^\alpha$ with $\alpha < 1$ and $\Phi(t) = t \log^{-\alpha}(e+t)$ with $\alpha > 1$.

Davis [Dav91] and Bass-Burdzy [BB92] studied IU for domains above the graph of a function. We write $x = (x', x_n) \in \mathbb{R}^n$. By $B'(x', R)$ we denote the $(n-1)$-dimensional open ball with center at x' and radius R.

Theorem B. *For a negative upper semicontinuous function $f(x')$ on $B'(0,R)$ we put*

$$D_f = \{(x', x_n) : |x'| < R,\ f(x') < x_n < 1\}.$$

Then we have the following assertions:

(i) *If $n = 2$ and $f \in L^\infty(B'(0,R))$, then D_f is IU (*[Dav91, Theorem 2]*).*
(ii) *If $f \in L^p(B'(0,R))$ with $p > n-1$, then D_f is IU (*[BB92, Theorem 1.22]*).*
(iii) *If $n \geq 3$, then there exists $f \in L^p(B'(0,R))$ with $p < n-1$ such that D_f is not IU (*[BB92, Section 4]*).*

Obviously, (i) is included in (ii). Note that D_f can be unbounded in (ii). We remark that D_f satisfies the quasihyperbolic boundary condition.

Proposition 2.16. *If $f \in L^p(B'(0,R))$ with $p > 0$, then D_f satisfies the QHB$((p+n-1)/p)$ condition.*

It is easy to see that $(p+n-1)/p < 2$ if and only if $p > n-1$. Hence, under the additional assumption of the CDC, Theorem B (ii) can be derived from Theorem A (i). The significance of Theorem B (ii) is that IU follows *without the CDC*. This remarkable phenomenon is rooted in [BB92, Lemma 2.4], which is reformulated analytically as an extended Harnack inequality with exceptional sets in [Aik14]. The critical case $p = n-1$ in Theorem B (iii) was open. Actually, we shall show in Corollary 2.21 below that there is $f \in L^{n-1}(B'(0,R))$ such that D_f is not IU in case $n \geq 3$. So, let us consider a condition sharper than $f \in L^{n-1}(B'(0,R))$. Let $\Phi(t)$ be a positive nondecreasing function of $t > 0$.

Theorem 2.17. *Assume that $\Phi(t)/t^{n-1}$ is nondecreasing and that*

$$\int_1^\infty \Phi(t)^{1/(1-n)} dt < \infty. \tag{2.3}$$

If $f \in L^\Phi(B'(0,R))$, then D_f is IU.

Remark 2.18. Typical examples of $\Phi(t)$ satisfying (2.3) are $\Phi(t) = t^p$ with $p > n-1$ and $\Phi(t) = t^{n-1}\log^\alpha(e+t)$ with $\alpha > n-1$.

Theorem 2.17 can be extended to L^Φ-domains. A counterpart of Theorem 2.17 for the GBHP has very different appearance.

Theorem C ([Aik14, Theorem 1.3])**.** *Let $\psi(t)$ be a nondecreasing continuous function for $t > 0$. Suppose that ψ satisfies* $\limsup_{t\to 0} \psi(Mt)/\psi(t) < M$ *for some $M > 1$ and*

$$\int_0^1 \frac{\psi(t)}{t} dt < \infty$$

Then every ψ-Hölder domain satisfies the GBHP.

Theorem 2.19. *Let $L > 0$ and let $r(t)$ be a positive nonincreasing L-Lipschitz function of $t \in [-1, \infty)$, i.e.,*

$$0 \leq r(t) - r(T) \leq L(T-t) \quad \text{for } -1 \leq t < T < \infty.$$

Define an infinite funnel *or a solid of rotation by*

$$V = \{(x', x_n) : -\infty < x_n < 1,\ |x'| < r(-x_n)\}.$$

See Figure 1. Then the following statements are equivalent:

(i) *V is IU.*

(ii) *V satisfies the Cranston-McConnell inequality.*

(iii) $\int_0^\infty r(t)dt < \infty.$

(iv) $\int_0^\infty w_\eta(k_V(\cdot, x_0) > s)^2 ds < \infty.$

(v) $\int_0^1 w_\eta(g < t)^2 \frac{dt}{t} < \infty.$

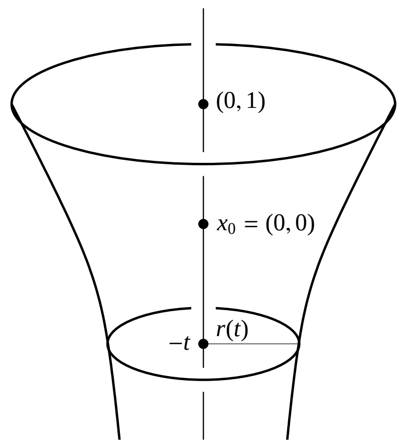

FIGURE 1. Infinite funnel.

Corollary 2.20. *Let $r(t) = (t+3)^{-1}$. Then V satisfies the QHB(2) and yet V is not IU.*

Corollary 2.21. *Let $n \ge 3$ and let $r(t) = (t+3)^{-1}\log^{-\alpha}(t+3)$ with $(n-1)^{-1} < \alpha \le 1$. Then V is not IU and yet V is represented as $D_f = \{(x', x_n) : |x'| < r(-1),\ x_n > f(x')\}$ with $f \in L^{n-1}(B'(0, r(-1)))$.*

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