Nonlinear Schrödinger equations describing energy dissipation and EDFA

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In this talk, we consider the Cauchy problem of the following nonlinear Schrödinger equation :

$$\begin{cases} i\partial_t u = -\dfrac{1}{2}\partial_x^2 u + \lambda|u|^{p-1}u, \\ u(0,x) = u_0(x), \end{cases} \tag{NLS}$$ eq:NLS

where $(t,x) \in \mathbb{R}\times\mathbb{R}$ and $u = u(t,x)$ is a complex valued unknown function. Unlike the usual nonlinear Schrödinger equation, we are going to deal with the coefficient λ as general complex number, i.e.,

$$\lambda = \lambda_1 + \lambda_2 \quad (\lambda_1, \lambda_2 \in \mathbb{R}).$$

The power of nonlinearity satisfies $1 < p$ at present, but it will be restricted in the assumption of the main results.

The NLS was proposed as a physical model, in the optical fiber engineering, governing the state of light signals propagating through an optical fiber. The real part of λ (described as λ_1) denotes the strength of nonlinear Kerr effect. The imaginary part of λ (described as λ_2) stands for the strength of nonlinear energy dissipation caused by impurities included in the optical fiber, if $\lambda_2 < 0$, and the strength of nonlinear amplification, if $\lambda_2 > 0$. In particular, such amplification is said to be caused by doped erbium in the fiber, and so it is called “ erbium doped fiber amplification (EDFA) ” .

Today, we are going to present two results concerning the behavior of the solutions. The first one has something to do with the optimal decay rate of large-data solution in the case of nonlinear energy dissipation. The second one states the existence of small-data-blow-up solution.

thm1 **Theorem 1** (Decay of the large-data solution : $\lambda_2 < 0$)**.** *Let* $\dfrac{7+\sqrt{193}}{8} < p \le 3$ *and* $\lambda_2 < 0$ *with* $(p-1)|\lambda_1| \le 2\sqrt{p}\,|\lambda_2|$*. Then, for any* $u_0 \in H^1$ *satisfying* $xu_0 \in L^2$ *(without size restriction), there exists a unique solution* u *to* (NLS) eq:NLS *such that*

(i) $u \in C([0,\infty); H^1)$ *and* $xu \in C([0,\infty); L^2)$,

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(ii)

$$\|u(t)\|_\infty \leq C \times \begin{cases} (1+t)^{-\frac{1}{2}}(\log(2+t))^{-\frac{1}{2}} & \textit{if} \quad p = 3, \\ (1+t)^{-\frac{1}{(p-1)}} & \textit{if} \quad \frac{7+\sqrt{193}}{8} < p < 3. \end{cases}$$

Theorem 1 is the minor progress in the sense that the range of p is slightly extended. In fact, Shimomura-Kita proved the corresponding consequence under $\dfrac{5+\sqrt{33}}{4} < p \leq 3$, where $\dfrac{5+\sqrt{33}}{4} \approx 2.68\cdots$ which is slightly greater than $\dfrac{7+\sqrt{193}}{8} \approx 2.61\cdots$. The idea of the proof is based on the decay estimate of " L^2-norm ", for which the error estimates are refined comparably to Shimomura-Kita ' s argument.

We also discuss the existence of small-data-blow-up solution in the case of EDFA. Note that, as for (NLS), there is no comparison principle like the nonlinear heat equations. It has been difficult to obtain a blow-up solution for small initial data.

thm2 **Theorem 2** (Existence of the small-data-blow-up solution : $\lambda_2 > 0$)**.** *Let $2 < p \leq 3$ and $\lambda_2 > 0$ with $(p-1)|\lambda_1| \leq 2\sqrt{p}\ |\lambda_2|$. Then, for any $\rho > 0$, there exists some $u_0 \in L^2$ such that,*

(i) $\|u_0\|_2 < \rho$,

(ii) the solution u to (NLS) with $u(0,x) = u_0(x)$ satisfies

$$\lim_{t \uparrow T^*} \|u(t)\|_2 = \infty,$$

for some $T^* > 0$.

Theorem 2 is proved as a corollary of Proposition 2.1 to be introduced in our talk. Unlike the nonlinear heat equations, the (NLS) is able to be solved in past direction. We apply this property to the construction of small-data-blow-up solution.