Stability of line solitons for KP-II

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The KP-II equation

$$\partial_x(\partial_t u + \partial_x^3 u + 3\partial_x(u^2)) + 3\sigma\partial_y^2 u = 0 \quad \text{for } t > 0 \text{ and } (x,y) \in \mathbb{R}^2 \quad \text{(KP)}$$

is a generalization to two spatial dimensions of the KdV equation

$$\partial_t u + \partial_x^3 u + 3\partial_x(u^2) = 0 \quad \text{for } t > 0 \text{ and } x \in \mathbb{R}. \tag{1}$$

Obviously, any solutions of (1) are solutions of (KP) which are uniform in y. The KP equation takes slow variations of the transversal direction into account. If $\sigma = 1$, then (KP) is called KP-II and it describes the motion of shallow water waves with weak surface tension.

The KdV equation has a two parameter family of solitary wave solutions

$$\{\varphi_c(x - 2ct - \gamma) \mid c > 0\,,\gamma \in \mathbb{R}\}\,, \quad \varphi_c(x) = c\operatorname{sech}^2\left(\sqrt{\frac{c}{2}}\,x\right).$$

The KP-II equation is expected to explain stability of these solitary wave solutions to the transversal perturbations (see [7]). When $\varphi_c(x - 2ct)$ is considered as a solution of the KP equations, we call it a *line soliton*. In this talk, I will discuss stability of line soliton solutions for the KP-II equation. The KP-II equation is known to be well-posed in $L^2(\mathbb{R}^2)$ on the background of line solitons ([11]).

The KP equation (KP) is an integrable system as well as the KdV equation and have conserved quantities such as

$$\int_{\mathbb{R}^2} u(t,x,y)^2\,dxdy \quad \text{(momentum)},$$

$$\frac{1}{2}\int \left(u_x^2(t,x,y) - 3\sigma(\partial_x^{-1}\partial_y u(t,x,y))^2 - 2u^3(t,x,y)\right)dxdy \quad \text{(Hamiltonian)}.$$

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For the KP-I equation, that is (KP) with $\sigma = -1$, the first two terms of the Hamiltonian have the same sign and it was shown by [6, 14] that there exists a stable ground state for the KP-I equation, whereas the KP-II equation has no traveling wave solutions that belong to $L^2(\mathbb{R}^2)$ ([5]).

On the other hand, it is known that the line solitons of the KP-I equation are unstable ([16]) and line solitons of the KP-II equation are linearly stable. Because the quadratic part of the Hamiltonian of the KP-II equation, which is the dominant part, is indefinite, it is more natural to explain stability of line solitons by using propagation estimates such as [13] rather than by using variational arguments such as [3].

The main difference between stability analysis of the KdV 1-soliton or the stability of line soliton with the y-periodic boundary condition is that $\varphi_c(x - 2ct)$ does not have the finite L^2-mass because it is not localized in the y-direction. As a consequence, the linearized operator of the KP-II equation around the line soliton has a family of continuous eigenvalues converging to 0 (see e.g. [1, 2]) in exponentially weighted space, whereas 0 is an isolated eigenvalue of the linearized KdV operator around a solitary wave in exponentially weighted space.

Since we have continuous spectrum converging to 0, the modulations of the speed and the phase shift of line solitons for the KP-II equation cannot be described by ODEs as KdV ([13]) or the KP-II equation posed on $\mathbb{R}_x \times \mathbb{R}_y/(2\pi\mathbb{Z})$ ([10]).

I find that for the KP-II equation posed on $\mathbb{R}^2$, modulation of the speed parameter $c(t, y)$ and the phase shift $x(t, y)$ cannot be uniform in y and their long time behavior is described by the Burgers equation. Note that similar modulation equations have formally derived for a two spatial dimensional Boussinesq model ([12]).

My results are the following ([9]).

Theorem 1 *Let $c_0 > 0$ and $a \in (0, \sqrt{c_0/2})$. Then there exist positive constants ε_0 and C satisfying the following: if $u(0, x, y) = \varphi_{c_0}(x) + v_0(x, y)$, $v_0 \in H^1(\mathbb{R}^2)$ and $\varepsilon := \|e^{ax}v_0\|_{L^2(\mathbb{R}^2)} + \|e^{ax}v_0\|_{L_y^1 L_x^2} + \|v_0\|_{L^2(\mathbb{R}^2)} <$*

ε_0, then there exist C^1-functions $c(t, y)$ and $x(t, y)$ such that for $t \geq 0$,

$$\|u(t, x, y) - \varphi_{c(t,y)}(x - x(t, y))\|_{L^2(\mathbb{R}^2)} \leq C\varepsilon, \qquad (2)$$

$$\sup_{y \in \mathbb{R}}(|c(t, y) - c_0| + |x_y(t, y)|) \leq C\varepsilon(1 + t)^{-1/2}, \qquad (3)$$

$$\|x_t(t, \cdot) - 2c(t, \cdot)\|_{L^2} \leq C\varepsilon(1 + t)^{-3/4}, \qquad (4)$$

$$\left\|e^{ax}(u(t, x + x(t, y), y) - \varphi_{c(t,y)}(x))\right\|_{L^2} \leq C\varepsilon(1 + t)^{-3/4}. \qquad (5)$$

We find that $c(t, y)$ and $\partial_y x(t, y)$ behave like a self-similar solution of the Burgers equation around $y = \pm\sqrt{8c_0}t$.

Theorem 2 *Let $c_0 = 2$ and let v_0 and ε be the same as in Theorem 1. Then for any $R > 0$,*

$$\left\|\begin{pmatrix} c(t, \cdot) \\ x_y(t, \cdot) \end{pmatrix} - \begin{pmatrix} 2 & 2 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} u_B^+(t, y + 4t) \\ u_B^-(t, y - 4t) \end{pmatrix}\right\|_{L^2(|y\pm 4t| \leq R\sqrt{t})} = o(t^{-1/4})$$

as $t \to \infty$, where $u_B^\pm$ are self similar solutions of the Burgers equation

$$\partial_t u = 2\partial_y^2 u \pm 4\partial_y(u^2)$$

such that

$$u_B^\pm(t, y) = \frac{\pm m_\pm H_{2t}(y)}{2\left(1 + m_\pm \int_0^y H_{2t}(y_1)\, dy_1\right)}, \quad H_t(y) = (4\pi t)^{-1/2}e^{-y^2/4t},$$

and that $m_\pm$ are constants satisfying

$$\int_{\mathbb{R}} u_B^\pm(t, y)\, dy = \frac{1}{4}\int_{\mathbb{R}} c(0, y)\, dy + O(\varepsilon^2).$$

The KP-II equation (KP) is invariant under a change of variables

$$x \mapsto x + ky - 3k^2 t + \gamma \quad \text{and} \quad y \mapsto y - 6kt \quad \text{for any } k, \gamma \in \mathbb{R}, \qquad (6)$$

and has a 3-parameter family of line soliton solutions

$$\mathcal{A} = \{\varphi_c(x + ky - (2c + 3k^2)t + \gamma) \mid c > 0, \, k, \gamma \in \mathbb{R}\}.$$

Thanks to propagations of the local phase shifts along the crest of line solitons, the set $\mathcal{A}$ is not stable in $L^2(\mathbb{R}^2)$.

Theorem 3 *Let $c_0 > 0$. There exists a positive constant C such that for any $\varepsilon > 0$, there exists a solution of* (KP) *such that $\|u(0, x, y) - \varphi_{c_0}(x)\|_{L^2} < \varepsilon$ and*

$$\liminf_{t \to \infty} t^{-1/4} \inf_{v \in \mathcal{A}} \|u(t, x, y) - v\|_{L^2(\mathbb{R}^2)} \geq C\varepsilon \,.$$

I remark that similar problems have already been studied for nonlinear heat equations ([15, 8]) and also for kink solutions to a wave equation ([4]). However, we need some correction of the phase shift for the KP-II equation, which does not appear in these former results because the transversal dimension and the power of nonlinearity is lower in our problem.

[1] J. C. Alexander, R. L. Pego and R. L. Sachs, *On the transverse instability of solitary waves in the Kadomtsev-Petviashvili equation*, Phys. Lett. A **226** (1997), 187–192.

[2] S. P. Burtsev, *Damping of soliton oscillations in media with a negative dispersion law*, Sov. Phys. JETP **61** (1985).

[3] T. Cazenave and P. L. Lions, *Orbital stability of standing waves for some nonlinear Schrödinger equations*, Comm. Math. Phys. **85** (1982), 549–561.

[4] S. Cuccagna, *On asymptotic stability in 3D of kinks for the ϕ^4 model*, Trans. Amer. Math. Soc. **360** (2008), 2581-2614.

[5] A. de Bouard and Y. Martel, *Non existence of L^2-compact solutions of the Kadomtsev-Petviashvili II equation*, Math. Ann. **328** (2004) 525-544.

[6] A. Bouard and J. C. Saut, *Remarks on the stability of generalized KP solitary waves*, Mathematical problems in the theory of water waves, 75–84, Contemp. Math. **200**, Amer. Math. Soc., Providence, RI, 1996.

[7] B. B. Kadomtsev and V. I. Petviashvili, *On the stability of solitary waves in weakly dispersive media*, Sov. Phys. Dokl. **15** (1970), 539-541.

[8] T. Kapitula, *Multidimensional stability of planar traveling waves*, Trans. Amer. Math. Soc. **349** (1997), 257–269.

[9] T. Mizumachi, *Stability of line solitons for the KP-II equation in $\mathbb{R}^2$*, http://arxiv.org/abs/1303.3532

[10] T. Mizumachi and N. Tzvetkov, *Stability of the line soliton of the KP-II equation under periodic transverse perturbations*, Mathematische Annalen **352** (2012), 659–690.

[11] L. Molinet, J. C. Saut and N. Tzvetkov, *Global well-posedness for the KP-II equation on the background of a non-localized solution*, Ann. Inst. H. Poincaré Anal. Non Linéaire **28** (2011), 653–676.

[12] G. Pedersen, *Nonlinear modulations of solitary waves*, J. Fluid Mech. **267** (1994), 83–108.

[13] R. L. Pego and M. I. Weinstein, *Asymptotic stability of solitary waves*, Comm. Math. Phys. **164** (1994), 305–349.

[14] Y. Liu and X. P. Wang, *Nonlinear stability of solitary waves of a generalized Kadomtsev-Petviashvili equation*, Comm. Math. Phys. **183** (1997), 253–266.

[15] J. X. Xin, *Multidimensional stability of traveling waves in a bistable reaction-diffusion equation, I.* Comm. Partial Differential Equations **17** (1992), 1889–1899.

[16] V. Zakharov, *Instability and nonlinear oscillations of solitons*, JEPT Lett. **22**(1975), 172-173.