Geometric flows for quadrature surfaces

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One of the classical problems in potential theory is to specify a surface Γ for a prescribed electric charge density μ in such a way that the uniform distribution of electric charges on Γ produces the same potential (at least in a neighborhood of the infinity) as μ. To derive a mathematical formulation of the problem precisely, let E be the fundamental solution of $-\Delta$ in $\mathbb{R}^N$, i.e.,

$$E(x) := \begin{cases} -\dfrac{1}{2\pi}\log|x| & (N=2), \\ \dfrac{1}{N(N-2)\omega_N|x|^{N-2}} & (N\geq 3), \end{cases}$$

where ω_N is the area of the unit ball in $\mathbb{R}^N$, and let $\mathcal{H}^{N-1}\lfloor\Gamma$ denote the $(N-1)$-dimensional Hausdorff measure restricted to Γ. Then, the problem can be stated as follows: for a prescribed finite Radon measure μ with compact support in $\mathbb{R}^N$, find a $(N-1)$-dimensional closed surface Γ enclosing a bounded domain Ω such that $E*\mu = E*\mathcal{H}^{N-1}\lfloor\Gamma$ in $\mathbb{R}^N\setminus\overline{\Omega}$, i.e.,

$$\int E(x-y)\,d\mu(y) = \int_\Gamma E(x-y)\,d\mathcal{H}^{N-1}(y) \quad \left(x\in\mathbb{R}^N\setminus\overline{\Omega}\right). \tag{1}$$

It can be shown that (1) is equivalent to the identity

$$\int h\,d\mu = \int_\Gamma h\,d\mathcal{H}^{N-1} \tag{2}$$

holding for all harmonic functions h defined in a neighborhood of $\overline{\Omega}$.

Definition 1. *A closed surface Γ satisfying (2) is called a quadrature surface of μ for harmonic functions.*

The mean value property of harmonic functions implies that (2) is valid when $\mu = N\omega_N\delta_0$ and $\Gamma = \partial B(0,1)$, where δ_0 is the Dirac measure supported at the origin and $B(0,1)$ is the unit ball in $\mathbb{R}^N$. Thus, the identity (2) can be seen as a generalization of the mean value formula for harmonic functions.

The existence of a quadrature surface Γ of a prescribed μ has been studied by several authors with different approaches. Developing the idea of super/subsolutions of Beurling [2], Henrot [4] was able to prove that the existence of Γ is guaranteed when a supersolution and a subsolution are available. Gustafsson & Shahgholian [3] followed a variational approach developed by Alt & Caffarelli [1], namely, they consider the minimization problem for the functional

$$J(u) := \int_{\mathbb{R}^N} \left(|\nabla u|^2 - 2fu + \chi_{\{u>0\}} \right) \, dx,$$

and obtain the existence and regularity of a minimizer u. Then, u satisfies the Euler-Lagrange equation

$$-\Delta u = f \lfloor \Omega - \mathcal{H}^{N-1} \lfloor \partial\Omega, \qquad \Omega = \{u > 0\},$$

and the existence of such a u implies that $\Gamma = \partial\Omega$ is a quadrature surface of μ with $d\mu = f\,dx$.

However, as pointed out by Henrot [4], the uniqueness of a quadrature surface cannot hold in general. The collapse of uniqueness seems to indicate a bifurcation phenomenon of solutions to (2) with a parametrized measure $\mu = \mu(t)$. Hence, toward understanding of the uniqueness issue, we need to consider the corresponding family of surfaces $\Gamma = \Gamma(t)$. In this respect, it is natural to ask if there is an evolution equation describing the moving surfaces $\{\Gamma(t)\}_{t>0}$ such that each $\Gamma(t)$ is a quadrature surface of a given parametrized measure $\mu(t)$. As a matter of fact, when $\mu(t) = t\delta_0 + \chi_{\Omega(0)}$ and the identity (2) is replaced by

$$\int h \, d\mu = \int_\Omega h \, dx, \tag{3}$$

it is known that the Hele-Shaw flow, a model of interface dynamics in fluid mechanics, surprisingly, plays the desired role. Here, analogously, a domain Ω satisfying (3) is called a quadrature domain of μ. Hence, the investigation of the evolution of quadrature domains is reduced to that of the Hele-Shaw flow, and the latter has been successfully proceeded by complex analysis and the theory of partial differential equations.

We introduce the following geometric evolution equation:

$$\begin{aligned} &v_n = p \quad \text{for } x \in \partial\Omega(t), \\ &\quad \text{where } \begin{cases} -\Delta p = \mu & \text{for } x \in \Omega(t), \\ (N-1)Hp + \dfrac{\partial p}{\partial n} = 0 & \text{for } x \in \partial\Omega(t), \end{cases} \end{aligned} \tag{4}$$

where v_n is the growing speed of $\partial\Omega(t)$ in the outer normal direction and H is the mean curvature of $\partial\Omega(t)$. The following theorem shows that, as desired, for a given $\partial\Omega(0)$ as initial surface, the solution to (4) turns out to be a one-parameter family of quadrature surfaces. Moreover, we will see that (4) is the only possible flow having this property. Here, we call $\{\partial\Omega(t)\}_{0\le t<T}$ a $C^{3+\alpha}$ family of surfaces if each $\partial\Omega(t)$ is of $C^{3+\alpha}$ and its time derivative is of $C^{2+\alpha}$, namely, $\partial\Omega(t)$ can be locally represented as a graph of a function in the Hölder space $C^{3+\alpha}$ and its time derivative is in $C^{2+\alpha}$.

Theorem 2. *Let $\{\partial\Omega(t)\}_{0\le t<T}$ be a $C^{3+\alpha}$ family of surfaces, and assume that each $\partial\Omega(t)$ has positive mean curvature. Then, each $\partial\Omega(t)$ is a quadrature surface of $\mu(t) := t\mu + \mathcal{H}^{N-1}\lfloor\partial\Omega(0)$ if and only if $\{\partial\Omega(t)\}_{0\le t<T}$ is a solution to (4).*

At this point, we are led to a fundamental question: Does the equation (4) really possess a unique smooth solution? The following theorem affirmatively answers this question. Here, $\{\partial\Omega(t)\}_{0\leq t<T}$ is called an $h^{3+\alpha}$ solution if it is an $h^{3+\alpha}$ family of surfaces and satisfies (4), where $h^{3+\alpha}$ is the so-called little Hölder space and is defined as the closure of the Schwartz space $\mathcal{S}$ of rapidly decreasing functions in the topology of the Hölder space $C^{3+\alpha}$.

Theorem 3. *There exists a local-in-time unique* $h^{3+\alpha}$ *solution* $\{\partial\Omega(t)\}_{0\leq t<T}$ *to* (4) *for any* $h^{3+\alpha}$ *initial surface* $\partial\Omega(0)$ *with positive mean curvature.*

References

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