Null structure in a system of quadratic derivative nonlinear Schrödinger equations *

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We consider the initial value problem for the system of nonlinear Schrödinger equations

$$\left(i\partial_t + \frac{1}{2m_j}\Delta\right) u_j = F_j(u, \partial u), \qquad t > 0,\ x \in \mathbb{R}^d,\ j = 1, 2, 3, \tag{1}$$

where $i = \sqrt{-1}$, $\partial_t = \partial/\partial t$, $\partial = (\partial_{x_a})_{a=1,\dots,d}$ for $x = (x_a)_{a=1,\dots,d} \in \mathbb{R}^d$ with $\partial_{x_a} = \partial/\partial x_a$, $\Delta = \sum_{a=1}^d \partial_{x_a}^2$, and $u = (u_j(t,x))_{j=1,2,3}$ is a $\mathbb{C}^3$-valued unknown function, while $\partial u = (\partial_{x_a} u_j)_{a=1,\dots,d;j=1,2,3}$ is its first order derivatives with respect to x. The nonlinear term $F = (F_j(u, \partial u))_{j=1,2,3}$ is assumed to be of the form

$$\begin{cases} F_1(u, \partial u) = \displaystyle\sum_{|\alpha|,|\beta|\le 1} C_{1,\alpha,\beta}\, (\overline{\partial^\alpha u_2})(\partial^\beta u_3), \\ F_2(u, \partial u) = \displaystyle\sum_{|\alpha|,|\beta|\le 1} C_{2,\alpha,\beta}\, (\partial^\alpha u_3)(\overline{\partial^\beta u_1}), \\ F_3(u, \partial u) = \displaystyle\sum_{|\alpha|,|\beta|\le 1} C_{3,\alpha,\beta}\, (\partial^\alpha u_1)(\partial^\beta u_2) \end{cases} \tag{2}$$

with some complex constants $C_{k,\alpha,\beta}$, and m_1, m_2, m_3 are positive constants.

The system (1) appears in various physical settings (see, e.g., [1] and the references cited therein). If the derivatives are not included in F, this system reads

$$\begin{cases} \left(i\partial_t + \frac{1}{2m_1}\Delta\right) u_1 = \overline{u_2} u_3, \\ \left(i\partial_t + \frac{1}{2m_2}\Delta\right) u_2 = u_3 \overline{u_1}, \\ \left(i\partial_t + \frac{1}{2m_3}\Delta\right) u_3 = u_1 u_2. \end{cases} \tag{3}$$

Note that the two-component system

$$\begin{cases} \left(i\partial_t + \frac{1}{2m_1}\Delta\right) u_1 = \overline{u_1} u_2, \\ \left(i\partial_t + \frac{1}{2m_2}\Delta\right) u_2 = {u_1}^2 \end{cases} \tag{4}$$

can be regarded as a degenerate case of (3). In the case of $d = 2$, Hayashi–Li–Naumkin [2] obtained a small data global existence result for (4) under the relation

$$m_2 = 2m_1. \tag{5}$$

The non-existence of the usual scattering state for (4) is also proved under (5). Higher dimensional case (i.e., $d \ge 3$) for (4) under the relation (5) is considered by Hayashi–Li–Ozawa [3] from the viewpoint of small data scattering. The above-mentioned results for the

*This talk is based on a joint work with Masahiro Ikeda and Soichiro Katayama [4].

two-component system (4) can be generalized to the three-component system (3) if the mass resonance relation (5) is replaced by

$$m_3 = m_1 + m_2. \tag{6}$$

However, it is non-trivial at all whether or not these can be generalized to the case of (1) under (6), because the presence of the derivatives in the nonlinearity causes a derivative loss in general. On the other hand, the presence of the derivatives in the nonlinearity sometimes yields extra-decay property. One of the most successful example will be the null condition introduced and developed by Christodoulou, Klainerman, John, Hörmander, Lindblad, Alinhac, etc. in the case of quadratic quasilinear systems of wave equations in three space dimensions.

The aim of this talk is to discuss analogous null structure in the derivative nonlinear Schrödinger system (1) with (2) under the mass resonance relation (6). We will introduce a structural condition on the nonlinearity under which the solution exists globally and it behaves like a free solution as $t \to \infty$ if the initial data are sufficiently small in a suitable weighted Sobolev space. Of particular interest is the case of $d = 2$, because the two-dimensional case for the Schrödinger equations corresponds to the three-dimensional case for wave equations from the viewpoint of the decay rate of solutions to the linearized equations.

Some related results ([5], [6]) will be also presented if time permits.

References

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