On the Cauchy problem for nonlinear Klein-Gordon equations in de Sitter spacetime

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We consider local and global energy solutions for the Cauchy problem of nonlinear Klein-Gordon equations in de Sitter spacetime. Let $n \geq 1$, $M > 0$, $H > 0$, $c > 0$, and let us consider the Cauchy problem given by

$$\begin{cases} (\partial_t^2 - c^2 e^{-2Ht}\Delta + M^2)u(t,x) + c^2 e^{nHt/2} f(e^{-nHt/2}u(t,x)) = 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for} \quad (t,x) \in [0,T) \times \mathbb{R}^n \\ u(0,\cdot) = u_0(\cdot) \in H^1(\mathbb{R}^n), \quad \partial_t u(0,\cdot) = u_1(\cdot) \in L^2(\mathbb{R}^n), \end{cases} \tag{0.1}$$

where u_0, u_1, f are real-valued functions, $\Delta := \sum_{j=1}^n \partial^2/\partial x_j^2$, $H^1(\mathbb{R}^n)$ denotes the Sobolev space and $L^2(\mathbb{R}^n)$ denotes the Lebesgue space. The spatial expansion of the de Sitter metric yields the dissipative effect to the Klein-Gordon equation.

D'Ancona and Giuseppe have shown in [5] and [6] global classical solutions for $(\partial_t^2 - a(t)\Delta)u + |u|^{p-1}u = 0$ with some additional conditions on $a(t) \geq 0$ and p when $n = 1, 2, 3$. Yagdjian has shown in [18] small global solutions for the first equation in (0.1) when the nonlinear term f is a power type of $f(u) = \pm|u|^{p-1}u$ or $\pm|u|^p$, $1 < p < \infty$, and the norm of initial data $\|u_0\|_{H^s(\mathbb{R}^n)} + \|u_1\|_{H^s(\mathbb{R}^n)}$ is sufficiently small for some $s > n/2 \geq 1$ (see also [19] for the system of the equations). Baskin has shown in [3] small global solution for $(\Box_g + \lambda)u + f(u) = 0$ when $f(u)$ is a type of $|u|^{p-1}u$, $p = 1 + 4/(n-1)$, $\lambda > n^2/4$, $(u_0, u_1) \in H^1 \oplus L^2$, where g gives the asymptotic de Sitter spacetime (see also [2] for the cases $p = 5$ with $n = 3$, $p = 3$ with $n = 4$). Blow-up phenomena are considered in [17]. See also the references in the summary [20] by Yagdjian. The aim of this paper is to give the fundamental theory for the well-posedness of the Cauchy problem (0.1) with power type nonlinear terms in the energy space, and we also consider exponential type nonlinear terms for the limiting case in terms of Sobolev embeddings in two spatial dimensions.

To denote power type nonlinear terms of order p, we define the following set $N(p)$. We note that the nonlinear terms $f(u) = \lambda|u|^{p-1}u$ and $f(u) = \lambda|u|^p$ for $\lambda \in \mathbb{R}$ satisfy $f \in N(p)$.

Definition 0.1 *Let $p \geq 1$. We denote by $N(p)$ the set of functions f from $\mathbb{R}$ to $\mathbb{R}$ which satisfies $f(0) = 0$ and*

$$|f(u) - f(v)| \leq C \max_{w=u,v} |w|^{p-1}|u - v| \tag{0.2}$$

for any u and $v \in \mathbb{R}$, where $C > 0$ is a constant independent of u and v.

For $T > 0$, we define a function space $X(T) := \{u : \|u\|_{X(T)} < \infty\}$, where

$$\begin{aligned} \|u\|_{X(T)} := \max\{&M\|u\|_{L^\infty((0,T),L^2(\mathbb{R}^n))}, \|\partial_t u\|_{L^\infty((0,T),L^2(\mathbb{R}^n))}, \\ &c\|e^{-Ht}\nabla u\|_{L^\infty((0,T),L^2(\mathbb{R}^n))}, c\sqrt{H}\|e^{-Ht}\nabla u\|_{L^2((0,T)\times\mathbb{R}^n)}\}. \end{aligned} \tag{0.3}$$

We start from the Cauchy problem for power type nonlinear terms.

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Theorem 0.2 *Let p satisfy*

$$1 \le p \begin{cases} < \infty & \text{if } \ n = 1, 2 \\ \le 1 + \frac{2}{n-2} & \text{if } \ n \ge 3. \end{cases} \tag{0.4}$$

Let $f \in N(p)$. Then we have the following results.

(1) For any u_0 and u_1, there exists $T = T(\|u_0\|_{H^1(\mathbb{R}^n)} + \|u_1\|_{L^2(\mathbb{R}^n)}) > 0$ such that (0.1) *has a unique solution u in $C([0,T), H^1(\mathbb{R}^n)) \cap C^1([0,T), L^2(\mathbb{R}^n))$. Here, u satisfies $u \in X(T)$, and for any fixed p_0 with $1 \le p_0 < 1 + 4/n$, there exists a constant $C > 0$ dependent on p_0 but independent of u_0 and u_1 such that T can be estimated from below as*

$$T \ge C\{\|u_0\|_{H^1(\mathbb{R}^n)} + \|u_1\|_{L^2(\mathbb{R}^n)}\}^{-(p-1)/\{1-n(p_0-1)/4\}}. \tag{0.5}$$

(2) If $\|u_0\|_{H^1(\mathbb{R}^n)} + \|u_1\|_{L^2(\mathbb{R}^n)}$ is sufficiently small, and $1 + 4/n \le p$, then (0.1) *has a unique solution u in $C([0,T), H^1(\mathbb{R}^n)) \cap C^1([0,T), L^2(\mathbb{R}^n))$. And u satisfies $u \in X(\infty)$.*

We use the following Gagliardo-Nirenberg interpolation inequality with asymptotic (see [10, Corollary 1.6], [11, Theorem 1.1] and the references therein) to consider exponential nonlinear terms.

Lemma 0.3 *Let $n = 1, 2$. There exist $\beta > 0$ and $q_0 \ge 2$ such that*

$$\|u\|_{L^q(\mathbb{R}^n)} \le \beta q^{1/2} \|\nabla u\|_{L^2(\mathbb{R}^n)}^{n(1/2-1/q)} \|u\|_{L^2(\mathbb{R}^n)}^{1-n(1/2-1/q)} \tag{0.6}$$

for any q with $q_0 \le q < \infty$ and nonconstant u. Here, β can be taken for any number with $\beta > (8\pi e)^{-1/2}$ when $n = 2$.

The exponential nonlinear terms have been considered for Schrödinger equations in [4, 13], wave equations in [7, 14], Klein-Gordon equations in [9, 15], heat equations in [8], complex Ginzburg-Landau equations and dissipative wave equations in [12], damped Klein-Gordon equations in [1]. We show the corresponding result for Klein-Gordon equations in de Sitter spacetime.

Theorem 0.4 *Let $n = 1, 2$. Let $\lambda \in \mathbb{R}$, $\alpha > 0$, $0 < \nu \le 2$, $j_0 \ge 0$. Let $f(u) = \lambda u(e^{\alpha|u|^\nu} - \sum_{0 \le j < j_0} \frac{\alpha^j}{j!} |u|^{\nu j})$ for $j_0 \ge 1$, and $f(u) = \lambda u e^{\alpha|u|^\nu}$ for $j_0 = 0$. Put $D := \|u_0\|_{H^1(\mathbb{R}^n)} + \|u_1\|_{L^2(\mathbb{R}^n)}$. Then we have the following results.*

(1) Let $\nu < 2$. For any u_0 and u_1, there exists $T > 0$ such that (0.1) *has a unique time local solution u in $C([0,T), H^1(\mathbb{R}^2)) \cap C^1([0,T), L^2(\mathbb{R}^2))$. Here, u satisfies $u \in X(T)$, and for any fixed p_0 with $j_0 \ge (p_0 - 1)/\nu$ and $1 \le p_0 < 1 + 4/n$, there exists a constant C_0 independent of D such that T can be estimated from below as*

$$T \ge \left(2C_0 \sum_{j \ge j_0} p(j) a(j) (2C_0 D)^{p(j)-1} \right)^{-1/\{1-n(p_0-1)/4\}}, \tag{0.7}$$

where $p(j) := \nu j + 1$, $a(j) := \alpha^j \beta^{p(j)} (2p(j))^{p(j)/2}/j!$, and β is any real number by which Lemma 0.3 holds.

(2) If $4/n\nu \le j_0$ and D is sufficiently small, then (0.1) *has a unique time global solution u in $C([0,\infty), H^1(\mathbb{R}^2)) \cap C^1([0,\infty), L^2(\mathbb{R}^2))$. And u satisfies $u \in X(\infty)$.*

We have considered the existence of solutions so far. Our solutions have the continuous dependence on the initial data, and they have asymptotic profiles to free solutions as follows.

Theorem 0.5 *Let u be the solution obtained in the above theorems for initial data u_0 and u_1, and let $0 < T \le \infty$ be the existence time of u there.*

(1) Let $v_0 \in H^1(\mathbb{R}^n)$ and $v_1 \in L^2(\mathbb{R}^n)$, and let v be the solution obtained in the above theorems for initial data v_0 and v_1. If v_0 converges to u_0 in $H^1(\mathbb{R}^n)$, and v_1 converges to u_1 in $L^2(\mathbb{R}^n)$, then $\|u - v\|_{X(T)}$ tends to zero.

(2) If u is the time global solution given by (2) of Theorem 0.2 and Theorem 0.4, then there exist $v_0 \in L^2(\mathbb{R}^n)$ and $v_1 \in H^{-1}(\mathbb{R}^n)$ such that

$$\lim_{t\to\infty}\{e^{-Ht}\|u(t) - v(t)\|_{L^2(\mathbb{R}^n)} + \|\partial_t u(t) - \partial_t v(t)\|_{H^{-1}(\mathbb{R}^n)}\} = 0, \tag{0.8}$$

where v is the free solution of $(\partial_t^2 - c^2 e^{-2Ht}\Delta + M^2)v = 0$, $v(0,\cdot) = v_0(\cdot)$, $\partial_t v(0,\cdot) = v_1(\cdot)$.

Finally, we consider global solutions for large data when the nonlinear term f in (0.1) has an energy conservative potential function.

Theorem 0.6 *Let $\lambda \ge 0$. Let $u_0 \in H^1(\mathbb{R}^n)$ and $u_1 \in L^2(\mathbb{R}^n)$. Let $f(u)$ be given by the following (1) or (2).*

(1) We put $f(u) = \lambda |u|^{p-1}u$, where p satisfies

$$1 \le p \begin{cases} < \infty & \text{if } n = 1, 2 \\ \le 1 + \frac{2}{n-2} & \text{if } n \ge 3. \end{cases} \tag{0.9}$$

(2) Let $n = 2$, $0 < \alpha < \infty$, $0 < \nu \le 2$, $0 \le j_0 < \infty$. Let $f(u) = \lambda u(e^{\alpha|u|^\nu} - \sum_{0\le j<j_0} \frac{\alpha^j}{j!}|u|^{\nu j})$ for $j_0 \ge 1$, and $f(u) = \lambda u e^{\alpha|u|^\nu}$ for $j_0 = 0$. When $\nu = 2$, we assume

$$\frac{1}{2}\int_{\mathbb{R}^2} c^2|\nabla u_0(x)|^2 + M^2 u_0^2(x) + |u_1(x)|^2 + c^2\lambda \sum_{j\ge j_0}^{\infty} \frac{\alpha^j}{j!2(j+1)}|u_0|^{2(j+1)}dx \le \frac{c^2\pi}{\alpha}. \tag{0.10}$$

Then (0.1) has a unique global solution u in $C([0,\infty), H^1(\mathbb{R}^n)) \cap C^1([0,\infty), L^2(\mathbb{R}^n))$. And u satisfies $u \in X(\infty)$.

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