GLOBAL EXISTENCE FOR NONLINEAR MASSLESS DIRAC EQUATIONS WITH NULL STRUCTURE IN 3D

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1. Introduction

In this note we consider the Cauchy problem for nonlinear massless Dirac equation:

$$\mathcal{D}\psi \equiv \gamma^0\partial_t + \gamma^j\partial_j\psi = F(\psi), \qquad (t,x) \in (0,\infty)\times\mathbf{R}^3, \tag{1.1}$$

$$\psi(0,x) = \psi_0(x), \qquad x \in \mathbf{R}^3 \tag{1.2}$$

where ψ is a $\mathbf{C}^4$-valued unknown function, $\partial_t = \partial_0 = \partial/\partial t$, $\partial_j = \partial/\partial x_j$ $(j=1,2,3)$, and γ^μ $(\mu = 0,1,2,3)$ is the Dirac matrices defined by

$$\gamma^0 = \begin{pmatrix} \sigma^0 & 0 \\ 0 & -\sigma^0 \end{pmatrix}, \quad \gamma^j = \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix} \ (j = 1,2,3)$$

with the Pauli matrices

$$\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We assume that ψ_0 is in a suitable weighted Sobolev space and its norm is small. We also assume that

$$F(\psi) = O(|\psi|^p), \tag{1.3}$$

around $\psi = 0$ for some $p > 1$.

It was shown by Tzvetokov [3] that if $p > 2$, then the problem for (1.1)-(1.2) admits a unique global solution for sufficiently small initial data, and that if $p = 2$, then such a global existence result does not hold for some nonlinearity. Nevertheless, it is conjectured in the same paper that a global solution to the problem exists for small initial data, if the nonlinearity takes such special forms as

$$F_0(\psi) = \langle \psi, \gamma^0\psi\rangle e \quad \text{or} \quad F_1(\psi) = \langle \psi, \gamma^0\gamma^5\psi\rangle e, \tag{1.4}$$

where e is a constant vector in $\mathbf{C}^4$, $\langle\ ,\ \rangle$ stands for the inner product in $\mathbf{C}^4$, and

$$\gamma^5 = \begin{pmatrix} 0 & \sigma^0 \\ \sigma^0 & 0 \end{pmatrix}.$$

The aim of this note is to give an affirmative answer to the above conjecture, by exploiting an additional decay from the special sesquilinear forms given by (1.4). Our result is as follows.

Theorem 1.1. *Assume that $\psi \in \mathcal{S}(\mathbf{R}^3)$ and that $F(\psi)$ is a linear combination of $F_0(\psi)$ and $F_1(\psi)$. Then there exists a positive number ε_0 such that for any $\varepsilon \in (0, \varepsilon_0)$, the Cauchy problem* (1.1)-(1.2) *has a unique classical solution ψ such that*

$$|\psi(t,x)| \le C(1+t+|x|)^{-1}(1+|t-|x||)^{-1}, \tag{1.5}$$

$$|(I + x_j|x|^{-1}\gamma^j\gamma^0)\psi(t,x)| \le C(1+t+|x|)^{-2}\log(2+t+|x|) \tag{1.6}$$

for $(t,x) \in [0,\infty) \times \mathbf{R}^3$.

Moreover, there exists a solution ψ_+ of $\mathcal{D}\psi_+ = 0$ such that

$$\lim_{t\to\infty} \|\psi(t) - \psi_+(t)\|_{L^2(\mathbf{R}^3)} = 0. \tag{1.7}$$

Remark. As is well known, we have $\mathcal{D}^2 = (\partial_t^2 - \Delta)I$. But, $\mathcal{D}F(\psi)$ does not satisfy the so-called null condition, so that the above theorem could not be deduced from the global existence result for the nonlinear wave equations due to [1], [2].

2. Null structure

For each fixed $\omega \in S^1$, we define

$$P_+ = I + \omega_j\gamma^j\gamma^0, \quad P_- = I - \omega_j\gamma^j\gamma^0.$$

Then a direct computation shows that

$$F_k(P_+\phi, P_+\psi) = F_k(P_-\phi, P_-\psi) = 0 \tag{2.1}$$

for any ϕ, $\psi \in \mathbf{C}^4$ and $k = 0, 1$. Since ϕ can be rewritten as

$$\phi = \frac{1}{2}(P_+\phi + P_-\phi),$$

we get

$$F_k(\phi,\psi) = \frac{1}{4}\left(F_k(P_+\phi, P_-\psi) + F_k(P_+\phi, P_-\psi)\right) \tag{2.2}$$

for any ϕ, $\psi \in \mathbf{C}^4$ and $k = 0, 1$.

3. Key lemma

Lemma 3.1. *Set $\Lambda_1 = \{(t,x) \in [0,\infty) \times \mathbf{R}^3 \,|\, 0 \le t \le 2|x|\}$. Assume that ψ solves* (1.1) *and satisfies*

$$|\psi(t,x)| \le C(1+t+|x|)^{-1}(1+|t-|x||)^{-1} \tag{3.1}$$

for $(t,x) \in \Lambda_1$. *Then we have*

$$|P_+\psi(t,x)| \le C(1+t+|x|)^{-2}\log(2+t+|x|) \tag{3.2}$$

for $(t,x) \in \Lambda_1$.

Once we find the improved estimate (3.2) for $P_+\psi$, we can derive an additional decay for the sesquilinear forms $F_0(\psi)$ and $F_1(\psi)$, in view of (2.2). Making use of this fact, one can prove Theorem 1.1 in a similar way to show the global existence result for nonlinear wave equations.

References

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