Critical exponent for the semilinear damped wave equation

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This talk is mainly based on the joint work [11] with M. Sobajima and Y. Wakasugi.
We consider the Cauchy problem for the semilinear damped wave equation

$$(P)_{c(t,x)} \qquad \begin{cases} u_{tt} - \Delta u + c(t,x)u_t = f(u) := |u|^p, & (t,x) \in \mathbf{R}_+ \times \mathbf{R}^N \\ (u, u_t)(0,x) = \varepsilon(u_0, u_1)(x), & x \in \mathbf{R}^N, \end{cases}$$

where $p > 1$ and the support of (u_0, u_1) is compact with $0 < \varepsilon \ll 1$. When the coefficient $c(t,x)$ of damping is

$$(1) \qquad c(t,x) = a(x)b(t) := a_0\langle x\rangle^{-\alpha}(1+t)^{-\beta}, \quad a_0 > 0, \ \langle x\rangle = \sqrt{1+|x|^2}, \ (\alpha, \beta \in \mathbf{R}),$$

our final aim is to obtain the critical exponent $p_c = p_c(N, \alpha, \beta)$ in the sense that, if $p > p_c$, then $(P)_{c(t,x)}$ has a global-in-time solution for small ε, and that, if $p \le p_c$, then the local-in-time solution blows up in a finite time for suitable data with any small ε.

When $c \equiv 1$ and $f(u) \equiv 0$, the solution u of our problem $(P)_{c=1}$ has the diffusion phenomenon, that is, the solution u behaves as the solution ϕ of corresponding parabolic problem

$$\phi_t - \Delta\phi = 0, \quad \phi(0,x) = \varepsilon(u_0 + u_1)(x)$$

(cf. Matsumura [8], Nishihara [10] etc.). In the result we can expect that the critical exponent p_c of $(P)_{c=1}$ with $f(u) = |u|^p$ equals to the Fujita exponent $p_F(N) = 1 + \frac{2}{N}$, which is, in fact, shown in Li-Zhou [5], Todorova-Yordanov [13], Zhang [16], Nishihara [10] etc.

We now consider $(P)_{c(t,x)}$ when $c(t,x)$ is given in (1). By the scaling invariant method, we can expect that, if $\alpha + \beta > 1$, then the damping is non-effective (cf. [9, 15]) and p_c is the Strauss exponent, and that, if $\alpha + \beta < 1$, then the damping is effective and p_c is the variant of $p_F(N)$. We mainly treat the case of effective damping. When the coefficient $c(t,x)$ depends only on the space x ($\beta = 0$), or time t ($\alpha = 0$), then we already have several results;

For $(P)_{c=\langle x\rangle^{-\alpha}}(0 \le \alpha < 1)$, $p_c(N, \alpha, 0) = 1 + \frac{2}{N-\alpha}$ (Ikehata-Todorova-Yordanov [4]),

For $(P)_{c=(1+t)^{-\beta}}(-1 < \beta < 1)$, $p_c(N, 0, \beta) = 1 + \frac{2}{N}$ (Lin-Nishihara-Zhai [7]).

Note that the case $\alpha = 1$ or $\beta = 1$ is delicate and the case $\beta < -1$ changes the situation. Though the small data global existence theorem is shown in [14, 6] etc. in the case $\alpha + \beta < 1$ with $\alpha \ge 0$, $\beta \ge 0$, the case $\alpha < 0$ was not known. Also, note that $p_c(N, 0, \beta)$ is independent of β. Thus, we treat the case $\alpha < 0$ and $\alpha + \beta < 1$. In fact, we obtain the following theorems in Nishihara-Sobajima-Wakasugi [11].

Theorem 1 (Global-in-time solution). *When $\alpha < 0$, $-1 < \beta < 1$ or $\alpha < 0$, $\beta = 1$ with $a_0 \gg 1$ in (1), if $1 + \frac{2}{N-\alpha} < p \le \frac{N}{[N-2]_+}$, then for small data $\varepsilon(u_0, u_1) \in H^1 \times L^2$, the Cauchy problem $(P)_{c(t,x)}$ admits a unique global-in-time solution $u \in C([0,\infty); H^1(\mathbf{R}^N)) \cap C^1([0,\infty); L^2(\mathbf{R}^N))$.*

Theorem 2 (Blow-up in finite time). *Assume $\alpha < 0$, $\beta = 0$ or $\alpha < 0$, $\beta = 1$ in (1). Then, if*

$$1 < p \le 1 + \frac{2}{N-\alpha} \quad \text{and} \quad \int_{\mathbf{R}^N} [(a_0\langle x\rangle^{-\alpha} - \beta)u_0(x) + u_1(x)]\,dx > 0,$$

then there is no global-in-time solution $u \in C([0,\infty);H^1)\cap C^1([0,\infty);L^2)$ *to* $(P)_{c(t,x)}$.

Moreover, the life-span of the solution

$$T_\varepsilon := \sup\{T;\ \textit{the solution}\, u \in C([0,T);H^1)\cap C^1([0,T);L^2)\ \textit{to}\ (P)_{c(t,x)}\ \textit{exists}\}$$

is estimated from above as

$$T_\varepsilon \le \begin{cases} C\varepsilon^{\frac{2-\alpha}{2(1+\beta)}\left(\frac{1}{p-1}-\frac{N-\alpha}{2}\right)^{-1}} & p < p_c(N,\alpha,1) \\ e^{C\varepsilon^{-(p-1)}} & p = p_c(N,\alpha,1). \end{cases}$$

To the small data global existence in the supercritical exponent we apply the weighted energy method, developed in [13] for the damped wave equations and new idea in [12], while, to the finite time blow-up we apply the test function method, developed in [13, 16], and [2] for the estimate of life-span. We remark that the critical exponent p_c is completely shown to be $p_c(N,\alpha,\beta) = 1 + \frac{2}{N-\alpha}$ when $\beta = 0$, $\alpha < 1$ or $\beta = 1$, $\alpha < 0$. When $\beta \ne 0$ and $\beta \ne 1$, the blow-up problem is still open.

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