LOCAL ENERGY WEAK SOLUTIONS FOR THE NAVIER-STOKES EQUATIONS AND APPLICATIONS

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1. Introduction

This talk is based on a joint work with Yasunori Maekawa (Kyoto university) and Christophe Prange (Université de Bordeaux, CNRS). We consider the Navier-Stokes equations in the half-space

$$\begin{cases} \partial_t u + u\cdot\nabla u - \Delta u + \nabla p = 0, \quad \nabla\cdot u = 0 & \text{in } (0,T)\times\mathbb{R}^3_+, \\ u = 0 & \text{on } (0,T)\times\partial\mathbb{R}^3_+, \end{cases} \tag{1.1}$$

for divergence-free initial data. There is a long history about finite energy weak solutions to (1.1) with initial data $u_0 \in L^2_\sigma(\Omega)$, where Ω can be for instance a bounded domain, $\mathbb{R}^3$ or $\mathbb{R}^3_+$, which goes back to the seminal works of Leray [10] and Hopf [5]. The study of weak solutions with infinite energy is much more recent. It is interesting in its own right since one can study richer dynamics generated by the solutions themselves and not driven by a source term. We are interested in a special kind of infinite energy solutions, so-called local energy weak solutions. For these solutions the energy is locally uniformly bounded. This notion of solutions has been pioneered by Lemarié-Rieusset [9] in the whole space $\mathbb{R}^3$, and later slightly extended by Kikuchi and Seregin [8]. Our goal is to extend the notion of solution to the half-space $\mathbb{R}^3_+$ and to prove global in time existence results. This answers an open problem mentioned by Barker and Seregin in [1, Section 1].

The class of local energy weak solutions, which will be made precise in Definition 2.1, is very useful, even for the study of finite energy weak solutions to (1.1), so-called Leray-Hopf solutions, for at least three reasons. The first reason is that they satisfy a local energy inequality. In particular, the solutions are suitable in the sense of Caffarelli, Kohn and Nirenberg [2, 11], so that we can apply ε-regularity theory to them. The second reason is that local energy weak solutions appear as limits of rescaled solutions of the Navier-Stokes equations. This is the case for instance when studying the local behavior of a Leray-Hopf solution near a potential singularity. The energy being supercritical in 3D with respect to the Navier-Stokes scaling $u_\lambda(y,s) = \lambda u(\lambda y, \lambda^2 s)$, the energy blows-up when zooming. The limit object is still a solution of the Navier-Stokes system, not in the finite energy class, but in the local energy class. Finally, the theory of local energy solutions plays also an important role in the seminal work of Jia and Šverák [6] about the construction of forward self-similar solutions with large initial data. This work and the subsequent studies [7, 4] represent a big progress toward understanding non-uniqueness of Leray-Hopf solutions.

Combining the features of the local energy weak solutions emphasized in the previous paragraph makes them powerful objects to study, for instance, blow-up of scale-critical norms near potential singularities. In this way, Seregin [13] was able to improve the celebrated result of Escauriaza, Seregin and Šverák [3]. Seregin proved that: if a weak finite energy solution u to (1.1) in $\mathbb{R}^3$ has a first singularity at time T, in the sense that u is smooth in the time interval $(0,T)$ and that the L^∞ norm of u is infinite in any parabolic cylinder $B(x_0,\rho)\times(T-\rho^2,T)$, for fixed $x_0\in\mathbb{R}^3$ and any $\rho>0$, then

$$\|u(\cdot,t)\|_{L^3(\mathbb{R}^3)}\longrightarrow\infty \qquad \text{as} \;\; t\to T-0.$$

One of our objectives is to show that the solutions we construct make it possible to prove the blow-up of the L^3 norm in the case of the half-space $\mathbb{R}^3_+$ following the scheme in [13]. Hence, we will recover the result of [1, Theorem 1.1] of the blow-up of the $L^{3,q}$ norm $3\le q<\infty$, in the case $q=3$.

2. Definition of local energy weak solutions and main results

Let us first recall the definition of loc-uniform Lebesgue spaces: for $1\le q\le\infty$,

$$L^q_{uloc}(\mathbb{R}^d_+):=\left\{f\in L^1_{loc}(\mathbb{R}^d_+)\;\Big|\;\sup_{\eta\in\mathbb{Z}^{d-1}\times\mathbb{Z}_{\ge0}}\|f\|_{L^q(\eta+(0,1)^d)}<\infty\right\}.$$

Let us define the space $L^p_{uloc,\sigma}(\mathbb{R}^d_+)$ of solenoidal vector fields in L^q_{uloc} as follows:

$$L^q_{uloc,\sigma}(\mathbb{R}^d_+):=\left\{f\in L^q_{uloc}(\mathbb{R}^d_+)^d\;\Big|\;\int_{\mathbb{R}^d_+}f\cdot\nabla\varphi\,dx=0\;\text{ for any }\varphi\in C_0^\infty(\overline{\mathbb{R}^d_+})\right\}.$$

For more properties of these spaces of locally uniformly p-integrable functions, see [12] and the references cited therein.

Here we state the definition of local energy weak solutions to (1.1) when the initial data belongs to

$$\mathcal{L}^2_{uloc,\sigma}(\mathbb{R}^3_+):=\overline{C^\infty_{c,\sigma}}^{L^2_{uloc}}(\mathbb{R}^3_+).$$

We will actually be able to construct local energy weak solutions for data in $L^2_{uloc,\sigma}(\mathbb{R}^3_+)$ locally in time. Nevertheless, the introduction of the space $\mathcal{L}^2_{uloc,\sigma}(\mathbb{R}^3_+)$ is useful since the solutions in this class decay at spatial infinity, and hence, the parasitic solutions (the flows driven by the pressure with linear growth) are automatically excluded in this class. Then we can state the definition of solutions in a simple fashion compared with the solutions in the class of nondecaying functions, where the structure of the pressure has to be included in the definition of solutions.

Definition 2.1. Let $T\in(0,\infty]$ and $Q_T:=(0,T)\times\mathbb{R}^3_+$. A pair (u,p) is called a local energy weak solution to (1.1) in Q_T with the initial data $u_0\in\mathcal{L}^2_{uloc,\sigma}(\mathbb{R}^3_+)$ if (u,p) satisfies the following conditions:

(i) We have $u \in L^\infty(0,T;\mathcal{L}^2_{uloc,\sigma}(\mathbb{R}^3_+))$ if $T<\infty$, $u \in L^\infty_{loc}([0,T);\mathcal{L}^2_{uloc,\sigma}(\mathbb{R}^3_+))$ if $T=\infty$ and $p \in L^{\frac{3}{2}}_{loc}((0,T)\times\overline{\mathbb{R}^3_+})$, and

$$\sup_{x\in\mathbb{R}^3_+}\int_0^{T'}\|\nabla u\|^2_{L^2(B(x)\cap\mathbb{R}^3_+)}dt+\sup_{x\in\mathbb{R}^3_+}\left(\int_\delta^{T'}\|\nabla p\|^{\frac{3}{2}}_{L^{\frac{9}{8}}(B(x)\cap\mathbb{R}^3_+)}dt\right)^{\frac{2}{3}}<\infty$$

for all finite $T'\in(0,T]$ and $\delta\in(0,T')$. Here $B(x)$ is the ball of radius 1 centered at x.

(ii) The pair (u,p) satisfies

$$\int_0^T -\langle u,\partial_t\varphi\rangle_{L^2(\mathbb{R}^3_+)}+\langle\nabla u,\nabla\varphi\rangle_{L^2(\mathbb{R}^3_+)}-\langle p,\operatorname{div}\varphi\rangle_{L^2(\mathbb{R}^3_+)}+\langle u\cdot\nabla u,\varphi\rangle_{L^2(\mathbb{R}^3_+)}\,dt=0$$

for any $\varphi\in C^\infty_c((0,T)\times\overline{\mathbb{R}^3_+})^3$ such that $\varphi|_{x_3=0}=0$.

(iii) The function $t\mapsto\langle u(t),w\rangle_{L^2(\mathbb{R}^3_+)}$ belongs to $C([0,T))$ for any compactly supported $w\in L^2(\mathbb{R}^3_+)^3$. Moreover, for any compact set $K\subseteq\overline{\mathbb{R}^3_+}$,

$$\lim_{t\to0}\|u(t)-u_0\|_{L^2(K)}=0.$$

(iv) The pair (u,p) satisfies the local energy inequality: for any $\chi\in C^\infty_c((0,T)\times\overline{\mathbb{R}^3_+})$ and for *a.e.* $t\in(0,T)$,

$$\begin{aligned}&\|(\chi u)(t)\|^2_{L^2(\mathbb{R}^3_+)}+2\int_0^t\|\chi\nabla u\|^2_{L^2(\mathbb{R}^3_+)}ds\\&\quad\le\int_0^t\langle|u|^2,\partial_s\chi^2+\Delta\chi^2\rangle_{L^2(\mathbb{R}^3_+)}+\langle u\cdot\nabla\chi^2,|u|^2+2p\rangle_{L^2(\mathbb{R}^3_+)}ds.\end{aligned}$$

The main result of our talk is stated as follows:

Theorem 2.2. *For any $u_0\in\mathcal{L}^2_{uloc,\sigma}(\mathbb{R}^3_+)$ there exists a local energy weak solution (u,p) to (1.1) in Q_∞ with initial data u_0.*

This result states the global in time existence of local energy weak solutions in the sense of Definition 2.1. It is the analog for the half-space of the theorem of Lemarié-Rieusset [9, Theorem 33.1] and of Kikuchi and Seregin [8, Theorem 1.5] for the whole space $\mathbb{R}^3$.

We next present an application of our result to a blow-up criteria in the half-space. Recall that a point (x_0,t) is called regular if u is bounded in a parabolic ball $B(x_0,r)\times(t-r^2,t)$. If (x_0,t) is not regular it is, by definition, singular. We say that u blows-up at time T if T is the time of the first occurrence of a singularity.

Corollary 2.3. *Let u be a finite energy weak solution (i.e. a Leray-Hopf solution) to the Navier-Stokes equations (1.1) with initial data $u_0\in L^2_\sigma(\mathbb{R}^3_+)$. Assume that u blows-up at a finite time $T>0$. Then*

$$\|u(\cdot,t)\|_{L^3(\mathbb{R}^3_+)}\longrightarrow\infty\qquad\textit{as}\quad t\to T-0.$$

This result is not new. It has been initially proved by Barker and Seregin in [1]. Our goal here is to give an alternative proof of this result, based on the existence theory of local energy weak solutions developed in our present work.

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