On dominating even subgraphs in cubic graphs

Roman Cada
University of West Bohemia
Univerzitní 8 306 14 Pilzeň, Czech Republic

Shuya Chiba*
Kumamoto University
Kumamoto 860-8555, Japan

Kenta Ozeki†
National Institute of Informatics
Tokyo 101-8430, Japan
and
JST, ERATO
Kawarabayashi Large Graph Project

Kiyoshi Yoshimoto‡
Nihon University
Tokyo 101-8308, Japan

September 12, 2015

Abstract
It is known that a 3-edge-connected graph has a spanning even subgraph in which every component contains at lest five vertices, and the lower bound is best possible. A natural question arises whether we can improve the lower bound by changing the spanning property with the dominating property. In this paper, we show that a 3-edge-connected cubic graph has a dominating even subgraph in which every component contains at least six vertices.

1 Introduction
In this paper, we consider finite graphs without loops. An even graph is a graph in which every vertex has a positive even degree and a subgraph $H$ of a graph $G$ is said to be dominating if $G - V(H)$ is edgeless. In this paper, a cycle is a connected 2-regular graph and a cycle with $l$ vertices is called an $l$-cycle. A 2-factor is a spanning 2-regular subgraph of a graph. An edge-cut is a minimal set of edges whose removal increase the number of components of the graph. We call an edge-cut with $l$ edges

*This work was supported by JSPS KAKENHI Grant Number 26800083.
†This work was in part supported by JSPS KAKENHI Grant Number 25871053 and by Grant for Basic Science Research Projects from The Sumitomo Foundation.
‡This work was supported by JSPS KAKENHI Grant Number 26400190.
an \( l \)-cut. An edge-cut is said to be essential if both of the two new components after deleting it have at least one edge.

For a vertex subset \( X \subset V(G) \), the set of edges joining \( X \) and \( V(G) - X \) is denoted by \( \partial(X) \) or simply \( \partial X \). If \( X \) consists of one vertex \( u \), then simply we denote \( \partial(u) \). For a subgraph \( H \) of \( G \), we use \( \partial H \) instead of \( \partial(V(H)) \). For terminology and notation not defined in this paper, we refer the readers to [4].

In this paper we consider a cubic graph, which is a 3-regular graph. A classical result by Petersen [17] is that a bridgeless cubic graph has a 2-factor. This well-known result was generalized by Fleischner [10] as follows: a bridgeless graph with minimum degree at least three has a spanning even subgraph in which every component has at least three vertices. If we restrict ourselves to simple graphs, then the lower bound of the order of components is improved to four [13]. Jackson and the fourth author considered 3-edge-connected graphs and showed the following.

**Theorem A** (Jackson and Yoshimoto [14]). A 3-edge-connected graph with \( n \) vertices has a spanning even subgraph in which each component contains at least \( \min\{5, n\} \) vertices.

They also gave an infinite family of 3-edge-connected cubic graphs in which every 2-factor contains 5-cycles. Thus the lower bound in the theorem is best possible in some sense. Kaiser and Škrekovski gave an interesting result, which also generalizes the Petersen’s theorem.

**Theorem B** (Kaiser and Škrekovski [15]). Every graph has an even subgraph which intersects all 3-cuts and 4-cuts.

If a given graph is bridgeless and cubic, then for any \( u \in V(G) \), \( \partial(u) \) is always minimal, i.e., it is a 3-cut of the graph, and so the above theorem implies the following.

**Corollary C.** A bridgeless cubic graph has a 2-factor which intersects all 3-cuts and 4-cuts.

If a given cubic graph is 3-edge-connected, then for an \( l \)-cycle \( C \) where \( l \in \{3, 4\} \), \( \partial C \) is an \( l \)-cut of the graph, and so Corollary C implies that a 3-edge-connected cubic graph has a 2-factor in which every component contains at least five vertices.

Matthews and Sumner [16] conjectured that 4-connected claw-free graphs are hamiltonian and Ryjáček [18] showed the Matthews-Sumner conjecture is equivalent
to the conjecture by Thomassen et al. [2, 6, 19] that 4-connected line graphs are hamiltonian. Fleischner and Jackson [11] showed that the conjecture on line graphs, and so the Matthews-Sumner conjecture, is equivalent to the conjecture by Ash and Jackson [3] that an essentially 4-edge-connected cubic graph has a dominating cycle. Thus it is interesting and important to study the behavior of dominating subgraphs in cubic graphs. In this paper, we prove the following using Corollary C.

**Theorem 1.** A 3-edge-connected cubic graph has a hamilton cycle or a dominating even subgraph $F$ such that every component in $F$ contains at least six vertices and $F$ intersects all essential 3-cuts.

In Section 2, we give several preparations for the proof of Theorem 1 and in Section 3, the proof will be given. Furthermore, we will give remarks on even subgraphs of 3-edge-connected cubic graphs and the Traveling salesman problem in Section 4.

We conjecture that Theorem 1 can be generalized as in Theorem A.

**Conjecture 1.** A 3-edge-connected graph with $n$ vertices has a dominating even subgraph in which each component has at least $\min\{6, n\}$ vertices.

Also it is a natural question to ask the upper bound of the lower bound of the order of components of dominating even subgraphs in 3-edge-connected graphs.

**Problem 2.** What is the upper bound $k$ such that any 3-edge-connected graph has a dominating even subgraph in which each component has at least $\min\{k, n\}$ vertices?

The following example implies the upper bound must be at most nine.

**Fact 1.** There is an infinite family of 3-edge-connected cubic graphs in which every dominating even subgraph has a cycle of order at most nine.

**Proof.** We construct such a cubic graph. Let $S$ be the graph as in Figure 1, where $S$ has 34 vertices, 49 edges and four “half-edges” whose one ends are in $S$. Later we define the other ends of the half-edges.

Let $m$ and $l$ be positive integers with $3m = 4l$. Let $B$ be $l$ copies of $S$ and $A$ be $m$ mutually disjoint triangles with three half-edges incident to each vertex of the triangle. See Figure 2. Since $A$ and $B$ have $3m$ and $4l$ half-edges, respectively, and $3m = 4l$, we can pair up half-edges in $A$ with half-edges in $B$. It is easy to pair up them such that the obtained graph $G$ is 3-edge-connected.
Figure 1:

Figure 2:
We show that any dominating even subgraph $F$ in $G$ has a cycle of order at most nine. If there is a triangle $T$ in $A$ such that $\partial T \cap F = \emptyset$, then obviously $F$ contains the 3-cycle $T$ as a component.

Suppose $\partial T \cap F \neq \emptyset$ for all triangle $T$ in $A$. Since $F$ is an even subgraph, $|\partial T \cap F| = 2$ for all triangle $T$ in $A$, and so $F$ contains $2m$ edges joining $A$ and $B$. Since $2m = 8l/3$, there is a component $S$ in $B$ such that $|\partial S \cap F| = 4$. Let $u_1u_2$ be the edge in the middle of $S$, see Figure 1, and let $L_S$ be the left component of $S - u_1u_2$. Let $v_1, v_2$ be the ends in $L_S$ of the edges in $\partial L_S - \{u_1u_2\}$.

Since $F$ is dominating $G$, at least one of the vertices $u_1$ and $u_2$ is contained in $F$, say $u_1$. Since $F$ is an even subgraph, $|\partial L_S \cap F| = 2$, and so $u_1u_2 \notin F$ and $\partial(u_1) - \{u_1u_2\} \subset F$. Since $\{v_1, v_2\} \subset F$ and $\partial(u_1) - \{u_1u_2\} \subset F$, the dominating even subgraph $F$ passes through every vertex in $L_S$. However, $L_S$ has no hamilton path joining $v_1$ and $v_2$. This fact forces that $F$ contains a cycle of length at most nine. □

The following question is also natural.

**Problem 3.** Does a 3-edge-connected cubic graph have a dominating even subgraph $F$ such that every component in $F$ contains at least six vertices and $F$ intersects all essential 3-cuts and 4-cuts?

## 2 Preparations

First, we give some additional notation. The set of all the neighbours of a vertex $x \in V(G)$ is denoted by $N_G(x)$ or simply $N(x)$, and its cardinality by $d_G(x)$ or $d(x)$. For a subgraph $H$ of $G$, we denote $N_G(x) \cap V(H)$ by $N_H(x)$ and its cardinality by $d_H(x)$. For simplicity, we denote $|V(H)|$ by $|H|$ and “$u_i \in V(H)$” by “$u_i \in H$”.

Similarly $G - V(H)$ is denoted by $G - H$.

An $i$-cell is the union of two 5-cycles in a cubic graph which have $i$ common edges. See Figure 3ab. We call a 5-cycle an $0$-cell. In the proof of Theorem 1, we will construct a dominating even subgraph from a 2-factor of a cubic graph which is obtained by reducing those cells. Hence we define reductions for those cells first.

Let $D$ be a 2-cell in $G$, and $u_1u_2 \cdots u_6u_1$ the 6-cycle and $w$ the remaining vertex in $D$. See Figure 4. Let $G'$ be the graph obtained from $G$ by contracting all of...
the paths $u_1u_6, u_2wv, u_3u_4$ and removing the edges $u_6u_5$ and $u_5u_4$. We denote this reduction $G' = G|D$.

Let $D$ be a 1-cell in $G$ and $u_1u_2 \cdots u_8u_1$ the 8-cycle of $D$. See Figure 5. Let $G'$ be the graph obtained from $G$ by removing the edge $u_2u_6$ and contracting both of the edges $u_1u_2$ and $u_7u_6$. We denote by $G' = G|D$ this reduction.

Let $D = u_1 \cdots u_5u_1$ be a 5-cycle without chord. Let $u'_i \in V(G - C)$ which is adjacent to $u_i$ for $1 \leq i \leq 5$. See Figure 6. Let $G'$ be the graph obtained from $G$ by removing the edges $u_1u_5, u_5u_4, u_4u_3$ and identifying $u_1, u_4$ and $u_3, u_5$, respectively. We denote by $G' = G|_{u_2}D$ this reduction.

We say a 5-cycle $C$ is good in $G$ if there is a 3-cut $T$ such that $|T \cap \partial C| \geq 2$. Notice that if $C$ has exactly one chord, then $C$ is always good since $\partial C$ is a 3-cut. If a 2- or 1-cell contains a good 5-cycle, then the cell is also called good. A cell which is not good is called bad. Notice that in a bad cell, every 5-cycle is bad.

We need the following fact in the proof of Theorem 1.
Fact 2. Let $i \in \{2, 1, 0\}$. If a 3-edge-connected cubic graph $G$ has a bad $i$-cell $D$, then $G|D$ or $G|_{u_2}D$ is 3-edge-connected.

This fact is obtained from the following two lemmas.

Lemma 3. Let $D = u_1u_2u_3u_4u_5u_1$ be a 5-cycle of a 3-edge-connected cubic graph. If there is a 3-cut $T$ such that $T \cap E(D) \neq \emptyset$, then $D$ is good.

Proof. Suppose $D$ is bad and there is a 3-cut $T$ such that $T \cap E(D) \neq \emptyset$. Since $D$ is bad and $G$ is 3-edge-connected, $D$ has no chord. Let $u'_i \in N_{G-C}(u_i)$ for $1 \leq i \leq 5$. Since $T$ is a minimum edge-cut of a cubic graph, no pair of edges in $T$ are adjacent, and so $T \cap E(D)$ contains independent two edges, say $u_1u_2, u_4u_5$. Then $(T - \{u_1u_2, u_4u_5\}) \cup \{u_1u'_1, u_5u'_5\}$ is a 3-cut containing two edges in $\partial C$, a contradiction. □

Lemma 4. For a $k(\geq 2)$-edge-connected cubic graph $G$, the following holds.

1. Let $D$ be a 2-cell and $u_1u_2\cdots u_6u_1$ be the 6-cycle in $D$. See Figure 4. If $G|D$ is not $k$-edge-connected, then $G$ has a $k$-cut containing $\{u_1u_2, u_5u_6\}$ or $\{u_2u_3, u_4u_5\}$.

2. Let $D$ be a 1-cell and $u_1u_2\cdots u_8u_1$ be the 8-cycle in $D$. See Figure 5. If $G|D$ is not $k$-edge-connected, then $G$ has a $k$-edge cut containing $\{u_1u_8, u_4u_5, u_2u_6\}$ or $\{u_3u_4, u_7u_8, u_2u_6\}$.

3. Let $D = u_1u_2\cdots u_5u_1$ be a 5-cycle and $u'_j$ be the vertex in $G - D$ which is adjacent to $u_j$ for $1 \leq j \leq 5$. See Figure 6. If $G|_{u_2}D$ is not $k$-edge-connected, then $G$ has a $k$-cut containing $\{u_1u'_1, u_4u'_4\}$ or $\{u_3u'_3, u_5u'_5\}$.

Proof. Let $G' = G|D$ or $G' = G|_{u_2}D$, $T$ be a minimum edge-cut of $G'$, and $D'$ be the subgraph in $G'$ corresponding to $D$. Let $S \subset V(G')$ such that $\partial S = T$ and $u_1 \in S$. 7
Suppose $|T| \leq k - 1$. Since $G$ is $k$-edge-connected, $T$ is not an edge-cut of $G$, and so $T = \partial S$ divides $D'$. For a vertex $u \in V(D)$, we denote a vertex in $G - D$ adjacent to $u$ by $u'$ if exists.

1. Since $|T \cap D'| = 1$, by symmetry, we may suppose $T \cap D' = \{u_1u_2\}$. Since $T$ is a minimum cut, no pairs of edges in $T$ are adjacent, and so $\{u'_1, u'_6\} \subset S$. Thus $\partial(S \cup \{u_6\})$ is a $k$-cut containing $\{u_1u_2, u_5u_6\}$ of $G$.

2. Since $T$ divides $V(D')$, $|T \cap D'| = 2$. By symmetry, we have four cases.

   a. If $T \cap D' = \{u_1u_3, u_5u_7\}$, then $\partial S$ is also a $(k - 1)$-cut of $G$, a contradiction.
   b. If $T \cap D' = \{u_1u_4, u_4u_5\}$, then $\partial(S \cup \{u_2, u_6\})$ is a $(k - 1)$-cut of $G$, a contradiction.
   c. If $T \cap D' = \{u_1u_6, u_6u_7\}$, then $\partial(S - \{u_1, u_3\})$ is a $(k - 1)$-cut of $G$, a contradiction.
   d. If $T \cap D' = \{u_1u_8, u_8u_5\}$, then since $\{u_1, u_3\} \subset S, \partial(S \cup \{u_2\})$ is a $k$-cut containing $\{u_1u_8, u_4u_5, u_2u_6\}$ of $G$.

3. Since $T = \partial S$ divides $D'$, $|T \cap D'| = 1$. By symmetry, we may suppose $T \cap D' = \{u_1u_2\}$. Since $\{u'_1, u'_4\} \subset S, \partial(S - u_1)$ is a $k$-cut containing $\{u_1u'_1, u_4u'_4\}$ of $G$. \[ \square \]

Proof of Fact 2. If $D$ is a 5-cycle and $G|_{u_2}D$ is not 3-edge-connected, then by Lemma 4, $D$ is good. If $D$ is a 2- or 1-cell and $G|D$ is not 3-edge-connected, then there exist a 5-cycle $C$ in $D$ and a 3-cut $T$ of $G$ such that $|D \cap T| \geq 2$ by Lemma 4. Thus by Lemma 3, $D$ is good. \[ \square \]

3 Proof of Theorem 1

Let $G$ be a 3-edge-connected cubic graph. We may assume $G$ is not hamiltonian; otherwise we are done. First we define a sequence of bad cells in $G$ which will be reduced.

Let 
$$
D_1 = \{D_1, D_2, \ldots, D_p\}
$$
be a maximal set of mutually disjoint 2-cells in $G$ such that $D_{i+1}$ is bad in $G_i = G_{i-1}|D_i$ for each $0 \leq i \leq p - 1$, where $G_0 = G$. If there is no bad 2-cell in $G$, then we define $D_1 = \emptyset$ and $p = 0$. We denote the subgraph in $G_i = G_{i-1}|D_i$ corresponding to $D_i$ by $D_i'$. See Figure 7. Notice that $G - \bigcup_{i \leq l} D_l = G_i - \bigcup_{l \leq i} D_i'$ and, by Fact 2, each $G_i$ is 3-edge-connected for any $0 \leq i \leq p$. By the maximality of $D_1$, obviously the following claim holds.
Claim 1. There is no 2-cell in $G - \bigcup_{i \leq p} D = G_p - \bigcup_{i \leq p} D'_i$ which is bad in $G_p$.

Let

$$D_2 = \{D_{p+1}, D_{p+2}, \ldots, D_{p+q}\}$$

be a maximal set of mutually disjoint 1-cells in $G - \bigcup_{i \leq p} D_i$ such that $D_{i+1}$ is bad in $G_i = G_{i-1} - D_i$ for each $p \leq i \leq p + q - 1$. If there is no bad 1-cell in $G - \bigcup_{i \leq p} D_i$, then we define $D_2 = \emptyset$ and $q = 0$. The subgraph in $G_i = G_{i-1} - D_i$ corresponding to $D_i$ is denoted by $D'_i$. In this case also, $G - \bigcup_{i \leq i} D_i = G_i - \bigcup_{i \leq i} D'_i$ and, by Fact 2, each $G_i$ is 3-edge-connected for any $0 \leq i \leq p + q$. 

Claim 2. There is no 1-cell in $G - \bigcup_{i \leq p+q} D = G_{p+q} - \bigcup_{i \leq p+q} D'_i$ which is bad in $G_{p+q}$ and there is no 2-cell $C$ in $G_{p+j} - \bigcup_{i \leq p+j} D'_i$ which is bad in $G_{p+j}$ for any $0 \leq j \leq q$. 

Proof. By the maximality of $D_2$, we have the first statement. If there is $1 \leq j \leq q$ such that $G_{p+j} - \bigcup_{i \leq p+j} D'_i$ contains a 2-cell $C$ which is bad in $G_{p+j}$, then obviously $C$ is bad in $G_{p+j-1}$ also, and so $C$ is bad in $G_p$. This contradicts Claim 1. \qed

Let $D_0$ be a maximal set of mutually disjoint bad 5-cycles in $G - \bigcup_{i \leq p+q} D_i$. For $D_1 \cup D_2 \cup D_0$, we define an independent set $R^*$ of $G$, whose vertices may not be contained in a dominating even subgraph of $G$ which is constructed later.

First, for each cell $D_i$ in $D_1 \cup D_2 \cup D_0$, we define pairs of vertices in $D_i$.

1. Let $D_i$ be a 2-cell in $D_1$ and $u_1 u_2 u_3 u_4 u_5 u_6 u_1$ be the 6-cycle in $D_i$. See Figure 3a. The pairs of $D_i$ are $\{u_1, u_3\}$ and $\{u_4, u_6\}$. 

![Figure 7:](image-url)
2. Let $D_i$ be a 1-cell in $D_2$ and $u_1u_2 \cdots u_8u_1$ be the 8-cycle in $D_i$. See Figure 3b.
   We define the pair of $D_i$ by $\{u_8, u_1\}$.

3. For a 5-cycle $D_i = u_1u_2 \cdots u_5u_1$ in $D_0$, the pair is defined by arbitrary two adjacent vertices in $D_i$, e.g., $\{u_1, u_2\}$. See Figure 3c.

Let $\mathcal{P}_0$ be the set of all the pairs for all $D_i \in D_1 \cup D_2 \cup D_0$. For each pair $\{u_i, u_j\} \in \mathcal{P}_0$, let $E_{u_i,u_j} = \partial(\{u_i, u_j\}) \cap \partial D_i$, where $\{u_i, u_j\} \subset D_i \in D_1 \cup D_2 \cup D_0$. Obviously $0 \leq |E_{u_i,u_j}| \leq 2$. Let

$$\mathcal{P} = \{\{u_i, u_j\} \in \mathcal{P}_0 : |E_{u_i,u_j}| = 2\} \text{ and } \mathcal{Q} = \bigcup_{\{u_i, u_j\} \in \mathcal{P}} E_{u_i,u_j}.$$  

We define the bipartite graph $H$ on the partite sets $\mathcal{P}$ and $\mathcal{Q}$ by defining the adjacency relation so that $\{u_i, u_j\} \in \mathcal{P}$ and $e \in \mathcal{Q}$ are adjacent if and only if $e \in E_{u_i,u_j}$. Since each element in $\mathcal{Q}$ is adjacent to at most two pairs in $\mathcal{P}$, for any $S \subset \mathcal{P}$,

$$2|S| = |E_H(S, N(S))| \leq |E_H(N(S), \mathcal{P})| \leq 2|N(S)|.$$  

Thus by Hall’s theorem, there is a matching $M$ in $H$ covering $\mathcal{P}$. Let $\varphi : \mathcal{P} \to \mathcal{Q}$ be the injection defined by $M$, i.e., for each $\{u_i, u_j\} \in \mathcal{P}$, the pair is adjacent to $\varphi(\{u_i, u_j\}) \in \mathcal{Q}$ by $M$. Let

$$R^* = \{u_k : u_k \in \{u_i, u_j\} \text{ is the end of the edge } \varphi(\{u_i, u_j\}) \text{ for some } \{u_i, u_j\} \in \mathcal{P}\}.$$  

Notice that $R^*$ is an independent set in $G$ since $\varphi$ is an injection.

Let

$$D_3 = \{D_{p+q+1}, D_{p+q+2}, \ldots, D_{p+q+r}\} \subset D_0$$

be a maximal subset of $D_0$ such that for $p + q \leq i \leq p + q + r - 1$,

1. $D_{i+1}$ is bad in $G_i = G_{i-1}|_{u_iD_i}$ where $u_i \in R^* \cap D_i$ and
2. $D_{i+1}$ intersects neither 3-cycle nor 4-cycle in $G_i$.

If there is no such 5-cycle, then we define $D_3 = \emptyset$ and $r = 0$. We denote by $D'_i$ the subgraph in $G_i = G_{i-1}|_{u_iD_i}$ corresponding to $D_i \in D_3$. Notice that each $D_i$ has no chord since $D_i$ is bad in $G_{i-1}$ and each $G_i$ is 3-edge-connected for any $p + q \leq i \leq p + q + r$ by Fact 2.
Claim 3. If there exist $0 \leq j \leq r - 1$ and $i \in \{2, 1, 0\}$ such that $G_{p+q+j} - \bigcup_{i \leq p+q+j} D'_i - D_{p+q+j+1}$ contains an $i$-cell $C$ which is good in $G_{p+q+j}$, then $C$ is good in $G_{p+q+j+1}$ also. 

Proof. Since $C$ is good in $G_{p+q+j}$, there exist a 3-cut $T$ of $G_{p+q+j}$ and a 5-cycle $C_1$ in $C$ such that $|T \cap \partial C_1| \geq 2$. Let $H_1$ and $H_2$ be the two components in $G_{p+q+j} - T$. Since $C \subseteq G_{p+q+j} - \bigcup_{i \leq p+q+j} D'_i - D_{p+q+j+1}$, the 5-cycle $D_{p+q+j+1}$ is contained in $H_1 - C$ or $H_2 - C$, and so both of $T$ and $C$ exits in $G_{p+q+r+1}$ also. Thus $C$ is good in $G_{p+q+r+1}$.

Claim 4. 
1. If $G_{p+q+j} - \bigcup_{i \leq p+q+j} D'_i$ has a bad 5-cycle, then the 5-cycle is bad in $G_{p+q}$ also.

2. There is neither 2- nor 1-cell $C$ in $G_{p+q+j} - \bigcup_{i \leq p+q+j} D'_i$ which is bad in $G_{p+q+j}$ for any $0 \leq j \leq r$.

Proof. Claim 3 implies the first statement immediately. By Claim 3, if there is $1 \leq j \leq r$ such that $G_{p+q+j} - \bigcup_{i \leq p+q+j} D'_i$ contains a 2- or 1-cell $C$ which is bad in $G_{p+q+j}$, then $C$ is bad in $G_{p+q+j-1}$, and so $C$ is bad in $G_{p+q}$. This contradicts Claim 2.

Let 

$$S_0 = \emptyset \text{ and } S_i = \bigcup_{1 \leq l \leq i} D'_l$$

for $1 \leq i \leq p + q + r$. We call a vertex in $S_i$ an yellow vertex.

We extend $R^*$ by $D_i \in D_1$ and $u_1 u_2 u_3 u_4 u_5 u_6 u_1$ be the 6-cycle. See Figure 3a. We define $R(D_i) = \{u_2, u_5\}$. Let $D_i \in D_2$ and $u_1 u_2 u_3 u_4 u_5 u_6 u_7 u_8 u_1$ be the 8-cycle. See Figure 3b. We define $R(D_i) = \{u_2, u_6\}$. Let 

$$R_0 = R^* \cup \bigcup_{1 \leq l \leq p+q} R(D_i) \subset V(G)$$

and 

$$R_i = R_0 - \bigcup_{1 \leq l \leq i} V(D_l) \subset V(G_i)$$

for $1 \leq i \leq p + q + r$. We call a vertex in $R_i$ a red vertex. By the definition of $S_i$ and $R_i$, there is no vertex in $G_i$ which is red and yellow. Notice that 

$$R_i = \begin{cases} 
R_{i+1} \cup (R^* \cap D_{i+1}) \cup R(D_{i+1}) & \text{ for } 0 \leq i \leq p + q - 1 \\
R_{i+1} \cup (R^* \cap D_{i+1}) & \text{ for } p + q \leq i \leq p + q + r - 1 
\end{cases}$$

(1)
For Theorem 1, it is enough to show that for all $0 \leq i \leq p + q + r$, $G_i$ has an even subgraph $F_i$ such that

1. $F_i$ intersects all 3-cuts in $G_i$.

2. Every component of $F_i$ contains at least five vertices. Especially if $F_i$ contains a 5-cycle $C$, then $C$ contains a yellow vertex, i.e., $C \cap S_i \neq \emptyset$.

3. Every vertex in $G_i - F_i$ is red, i.e., $G_i - F_i \subseteq R_i$, and $|D_i - F_i| \leq 1$ for $i + 1 \leq l \leq p + q + r$.

Indeed by the second condition, any component in $F_0$ contains at least six vertices as $S_0 = \emptyset$. Although $\bigcup_{i \leq p+q} R(D_i) \subseteq R_0$ may not be independent, since $|D_l - F_0| \leq 1$ for all $1 \leq l \leq p + q + r$, we have $G_0 - F_0(\subseteq R_0)$ is independent, i.e., $F_0$ is dominating $G = G_0$. Therefore $F_0$ is a desired even subgraph.

We construct $F_i$ inductively. First we show the existence of $F_{p+q+r}$. Notice that $G_i$ is 3-edge-connected for every $0 \leq i \leq p + q + r$ by Fact 2 as we reduced bad cells.

**Claim 5.** There is a 2-factor $F_{p+q+r}$ in $G_{p+q+r}$ such that

1. $F_{p+q+r}$ intersects all 3-cuts and 4-cuts in $G_{p+q+r}$ and

2. a 5-cycle $C$ in $F_{p+q+r}$ contains a yellow vertex, i.e., $C \cap S_{p+q+r} \neq \emptyset$.

**Proof.** Since $G_{p+q+r}$ is a 3-edge-connected cubic graph, by Corollary C, $G_{p+q+r}$ has a 2-factor $F_{p+q+r}$ which intersects all 3-cuts and 4-cuts. We choose $F_{p+q+r}$ such that the number of components is as small as possible.

Suppose $F_{p+q+r}$ contains a 5-cycle $C$ without a yellow vertex. If $C$ is good in $G_{p+q+r}$, then there is a 3-cut $T$ such that $|T \cap \partial C| \geq 2$, and $F_{p+q+r}$ does not intersect the 3-cut $T$, a contradiction. Therefore $C$ is bad in $G_{p+q+r}$. Since $C$ has no yellow vertex, $C \subseteq G_{p+q+r} - S_{p+q+r}$, and so the 5-cycle $C$ exists in $G_{p+q}$ and, by Claim 4, $C$ is bad in $G_{p+q}$ also.

Suppose $C \notin D_0$. By the maximality of $D_0$, there is a bad 5-cycle $D \in D_0$ intersecting $C$. If $|E(C \cap D)| \leq 2$, then $C \cup D$ is a 2- or 1-cell in $G_{p+q}$. Since both of $C$ and $D$ are bad in $G_{p+q}$, $C \cup D$ is bad in $G_{p+q}$. This contradicts Claim 2.

If $|E(C \cap D)| = 3$, then $D$ is a 5-cycle in $G_{p+q+r}$ also. However, $F_{p+q+r}$ does not contain the vertex in $D - C$ as $C$ is a component of $F_{p+q+r}$. This is a contradiction.
Therefore \( C \in D_0 \). Since \( C \) is bad in \( G_{p+q+r} \) and \( C \notin D_3 \), \( C \) intersects a 3- or a 4-cycle \( C_1 \) in \( G_{p+q+r} \). Since \( C \) is bad, \( C \) has no chord, and so \( C_1 - C \neq \emptyset \). If \( C_1 - C \) is a vertex \( w \), then \( w \) is not contained in \( F_{p+q+r} \) as \( C \) is a component of the 2-factor \( F_{p+q+r} \). This is a contradiction.

If \( C_1 - C \) contains an edge \( ww' \), then there is a component \( C_2 \) in \( F_{p+q+r} \) containing the edge \( ww' \). Since the symmetric difference \( C = C_1 \triangle C_2 \) is a cycle, the subgraph \( (F_{p+q+r} - C \cup C_2) \cup \tilde{C} \) is a 2-factor of \( G_{p+q+r} \) in which the number of components is less than \( F_{p+q+r} \). This contradicts the choice of \( F_{p+q+r} \). Thus \( C \) contains a yellow vertex. 

Suppose \( G_{i+1} \) has a desired even subgraph \( F_{i+1} \) for \( 1 \leq i + 1 \leq p + q + r \). Since \( G_{i+1} \) has no vertex which is yellow and red and \( F_{i+1} \) contains every vertex which is not red in \( G_{i+1} \), \( \bigcup_{1 \leq i+1} D_i' \subset F_{i+1} \).

**Claim 6.** If \( F_i \) is an even subgraph of \( G_i \) obtained from \( F_{i+1} \) by replacing edges in \( D_i' \) with edges in \( D_i+1 \), i.e.,

\[
E(F_{i+1}) - E(D_i') = E(F_i) - E(D_{i+1}),
\]

then the following holds.

1. \( F_i \) intersects all 3-cuts in \( G_i \).

2. Every component \( C \) of \( F_i \) intersecting no edge of \( D_{i+1} \) contains at least five vertices. Especially if \( C \) is a 5-cycle, then \( C \) contains a yellow vertex, i.e., \( C \cap S_i \neq \emptyset \).

3. Every vertex in \( G_i - F_i - D_{i+1} \) is red, i.e., \( G_i - F_i - D_{i+1} \subset R_i \), and \( |D_j - F_i| \leq 1 \) for \( j \geq i + 2 \).

**Proof.**
1. Let \( T \) be any 3-cut of \( G_i \). Since \( D_{i+1} \) is bad in \( G_i \), \( T \cap E(D_{i+1}) = \emptyset \) by Lemma 3. This implies \( T \) is a 3-cut of \( G_{i+1} \) and \( T \cap E(D_i') = \emptyset \). Thus \( F_{i+1} - E(D_i') = F_i - E(D_{i+1}) \) intersects \( T \).

2. Obviously \( C \) is a component of \( F_{i+1} \) also. Thus \( |C| \geq 5 \) and \( C \) contains a yellow vertex in \( S_{i+1} - V(D_i') = S_i \) if \( |C| = 5 \).
3. Since $G_i - D_{i+1} = G_{i+1} - D'_{i+1}$, we have $G_i - D_{i+1} - F_i = G_{i+1} - D'_{i+1} - F_{i+1} \subset R_{i+1} \subset R_i$ by (1). For $j \geq i + 2$,

$$D_j - F_i = D_j - F_{i+1},$$

and so we have $|D_j - F_i| = |D_j - F_{i+1}| \leq 1$. \hfill \Box

In the remaining part of this paper, we will construct a desired even subgraph $F_i$ of $G_i$ from $F_{i+1}$ by replacing edges in $D'_{i+1}$ with edges in $D_{i+1}$. From the above claim, it is enough to show that $F_i$ satisfies the following:

A1. Every component $C$ containing an edge of $D_{i+1}$ in $F_i$ contains at least five vertices. Especially if $C$ is a 5-cycle, then $C$ contains a yellow vertex, i.e., $C \cap S_i \neq \emptyset$.

A2. A vertex in $D_{i+1} - F_i$ is red, i.e., $D_{i+1} - F_i \subset R_i$, and $|D_{i+1} - F_i| \leq 1$.

We divide our argument into the following three cases.

1. $0 \leq i \leq p - 1$.

2. $p + q \leq i \leq p + q + r - 1$.

3. $p \leq i \leq p + q - 1$.

The first case is easier than the other cases. If there is a vertex in $G_i - D_{i+1}$ which is adjacent to $u \in D_{i+1}$, then we denote the vertex by $u'$.

1. $0 \leq i \leq p - 1$, i.e., $D_{i+1} \in D_1$.

Since $V(D'_{i+1}) \subset F_{i+1}$ and $F_{i+1}$ is an even subgraph, $|F_{i+1} \cap \partial D'_{i+1}|$ is 2 or 4. If the subgraph induced by $V(D_{i+1})$ contains an edge that is not in $E(D_{i+1})$, then $|\partial D_{i+1}| = 3$. This implies $D_{i+1}$ contains a good 5-cycle, i.e., $D_{i+1}$ is good in $G_i$. This contradicts our assumption. Therefore, both of $\{u_1, u_3\}$ and $\{u_4, u_6\}$ contain a red vertex.

Case 1. $|F_{i+1} \cap \partial D'_{i+1}| = 4$.

Since $D'_{i+1} \subset F_{i+1}$, by symmetry, we may suppose

$$F_{i+1} \cap \partial D'_{i+1} = \{u_1u'_6, u_1u'_1, u_2w', u_3u'_3\},$$
and then $F_{i+1}$ contains the edge $u_2u_3$. See Figure 8. Let $F'_i$ be the even subgraph obtained from $F_{i+1}$ by replacing $u'_1u_6$ and $w'u_2u_3'$ by $u'_1u_1u_6u'_6$ and $w'wu_5u_4u_3'u'_3$.

Since every component $C$ containing an edge of $D_{i+1}$ in $F_i$ contains at least six vertices, A1 holds. Since $u_2 \in R(D_{i+1}) \subset R_i$, A2 holds.

Case 2. $|F_{i+1} \cap \partial D'_{i+1}| = 2$. 

Since $D'_{i+1} \subset F_{i+1}$, $F_{i+1} \cap \partial D'_{i+1}$ does not contain the edge $u_2'u_2'$. Thus by symmetry, we may suppose $F_{i+1} \cap \partial D'_{i+1}$ is $\{u_1'u_6', u_3'u_3'\}$ or $\{u_1'u_6', u_3'u_4'\}$.

See Figure 9ab. If the intersection is $\{u_1'u_6', u_3'u_3'\}$, then the even subgraph $F_i$ obtained from $F_{i+1}$ by replacing $u_1u_2u_3$ by $u_6u_1u_2wu_5u_4u_3$ is a desired even subgraph because both of A1 and A2 hold as $V(D_{i+1}) \subset F_i$.

Suppose $F_{i+1} \cap \partial D'_{i+1} = \{u_1'u_6', u_3'u_3'\}$. For the pair $\{u_1, u_3\}$, if $u_3$ is red, i.e., $u_3 \in R_i$, then the even subgraph $F_i$ obtained from $F_{i+1}$ by replacing $u'_6u_1u_3'w'w_5u_4u_4'$. 

Figure 8:

Figure 9:
is a desired even subgraph because both of A1 and A2 hold. Similarly if \( u_1 \in R_i \), then the even subgraph \( F_i \) obtained from \( F_{i+1} \) by replacing
\[
u'_6 u_1 u_2 u_3 u'_4 \text{ by } u'_6 u_6 u_5 w u_2 u_3 u'_4 u'_4
\]
is a desired even subgraph.

2. \( p + q \leq i \leq p + q + r - 1 \), i.e., \( D_{i+1} \in D_3 \).

In this case, \( D_{i+1} = u_1 u_2 \cdots u_5 u_1 \) is a 5-cycle. By symmetry, we may suppose \( u_2 \) is red, i.e., \( u_2 \in R_i \). Since \( V(D_{i+1}') \subset F_{i+1} \), \( |F_{i+1} \cap \partial D_{i+1}'| \) is 2 or 4.

Case 1. \( |F_{i+1} \cap \partial D_{i+1}'| = 2 \).

Notice that \( F_{i+1} \cap \partial D_{i+1}' \) is not \( \{u_1 u'_4, u_2 u'_2\} \) because \( V(D_{i+1}') \subset F_{i+1} \). Hence by symmetry, we have the following three cases:

\( F_{i+1} \cap \partial D_{i+1}' \) is \( \{u_1 u'_4, u_3 u'_5\}, \{u_1 u'_4, u_3 u'_3\} \) or \( \{u_1 u'_1, u_3 u'_3\} \).

(i) Suppose \( F_{i+1} \cap \partial D_{i+1}' = \{u_1 u'_4, u_3 u'_5\} \). See Figure 10a. As \( V(D_{i+1}') \subset F_{i+1} \), \( F_{i+1} \)

contains the path \( u'_4 u_1 u_2 u_3 u'_5 \). Let \( F_i \) be the even subgraph in \( G_i \) which is obtained from \( F_{i+1} \) by replacing
\[
u'_4 u_1 u_2 u_3 u'_5 \text{ by } u'_4 u_4 u_3 u_2 u_1 u_5 u'_5.
\]

Obviously A1 holds. Since \( V(D_{i+1}) \subset F_i \), A2 holds.
(ii) Suppose $F_{i+1} \cap \partial D'_{i+1} = \{u_1u'_4, u_3u'_3\}$. See Figure 10b. Then $F_{i+1}$ contains the path $u'_4u_1u_2u_3u'_3$ as $V(D'_{i+1}) \subset F_{i+1}$. Let $F_i$ be the even subgraph in $G_i$ which is obtained from $F_{i+1}$ by replacing $u'_4u_1u_2u_3u'_3$ by $u'_4u_4u_5u_1u_2u_3u'_3$.

Obviously both of A1 and A2 hold.

(iii) Suppose $F_{i+1} \cap \partial D'_{i+1} = \{u_1u'_4, u_1u'_1, u_3u'_3, u_3u'_5\}$. See Figure 10c. Let $F_i$ be the even subgraph in $G_i$ which is obtained from $F_{i+1}$ by replacing $u'_1u_1u_2u_3u'_3$ by $u'_1u_1u_5u_4u_3u'_3$.

Since the component in $F_i$ containing an edge in $D_{i+1}$ contains at least six vertices, A1 holds. As $D_{i+1} - F_i = \{u_2\} \subset R_i$, A2 holds.

Case 2. $|F_{i+1} \cap \partial D'_{i+1}| = 4$.

As $V(D'_{i+1}) \subset F_{i+1}$, $F_{i+1} \cap \partial D'_{i+1}$ is not $\{u_1u'_4, u_1u'_1, u_3u'_3, u_3u'_5\}$. Thus by symmetry, we have two cases.

(i) Suppose $F_{i+1} \cap \partial D'_{i+1} = \{u_1u'_4, u_1u'_1, u_2u'_2, u_3u'_3\}$, and then $u_2u_3 \in F_{i+1}$. See Figure 11a. Let $F_i$ be the even subgraph in $G_i$ which is obtained from $F_{i+1}$ by replacing $u'_4u_1u_2u_3u'_3$ by $u'_4u_4u_5u_1u_2u_3u'_3$.

Since $V(D_{i+1}) \subset F_i$, A2 holds.

The component continuing $u'_4u_4u_5u_1u'_1$ of $F_i$ contains at least six vertices. Suppose $C = u_2u_3u'_3wu'_2u_2$ is a 5-cycle and $C \cap S_i = \emptyset$. See Figure 11b. Then
\( \tilde{C} = C \cup D_{i+1} \) is a 1-cell in \( G_i - \bigcup_{j \leq i} D_j' \). Since \( D_{i+1} \in D_3 \), \( D_{i+1} \) is bad in \( G_i \). Suppose that \( C \) is good and let \( T \) be a 3-cut such that \( |T \cap \partial C| \geq 2 \). Since \( D_{i+1} \) is bad, \( T \cap \partial C \subseteq \partial C - \{u_2u_1, u_3u_4\} \) by Lemma 3. Since \( T \) is a 3-cut of \( G_{i+1} \) also and \( F_{i+1} \) contains \( C \) as a component, \( F_{i+1} \) does not intersect \( T \), a contradiction. See Figure 11c. Thus both of \( D_{i+1} \) and \( C \) are bad, and so \( \tilde{C} \) is a bad 1-cell in \( G_i \). This contradicts Claim 4.

(ii) Suppose

\[
F_{i+1} \cap \partial D_{i+1}' = \{u_1u_4', u_1u_1', u_2u_2', u_3u_5'\},
\]

and then \( u_2u_3 \in F_{i+1} \). See Figure 12a. Let \( F_i \) be the even subgraph in \( G_i \) which is obtained from \( F_{i+1} \) by replacing

\[
u_4' u_1' \text{ and } u_2' u_3 u_5' \text{ by } u_4' u_4 u_3 u_2' \text{ and } u_1' u_1 u_5 u_5'.
\]

Since \( V(D_{i+1}) \subseteq F_i \), A2 holds.

Let \( C_1 \) and \( C_2 \) be the components in \( F_i \) containing \( u_1u_3 \) and \( u_2u_3u_4 \), respectively. Suppose \( C_1 \) or \( C_2 \) contains at most five vertices, and then \( C_1 \neq C_2 \). Since \( D_{i+1} \) intersects neither 3-cycle nor 4-cycle, \( C_1 \) or \( C_2 \) is a 5-cycle.

Suppose \( C_1 = u_1u_5u_1'wu_1u_1 \) is a 5-cycle and \( C_1 \cap S_i = \emptyset \). See Figure 12b. Then \( \bar{C}_1 = C_1 \cup D_{i+1} \) is a 1-cell in \( G_i \). By Claim 4, \( \bar{C}_1 \) is good. Since \( D_{i+1} \) is bad, \( C_1 \) is good, and so there is a 3-cut \( T \) in \( G_i \) such that \( |T \cap \partial C_1| \geq 2 \), and by Lemma 3 \( T \cap \partial C_1 \subseteq \partial C_1 - \{u_1u_2, u_5u_4\} \). Thus \( T \) is a 3-cut of \( G_{i+1} \). Since \( F_{i+1} \) contains the path \( u_1u_1'wu_1'\), \( F_{i+1} \) does not intersect \( T \), a contradiction. See Figure 12c.
Suppose $C_2 = u_2u_3u_4u'_5u'_2u_2$ is a 5-cycle and $C_2 \cap S_i = \emptyset$. See Figure 12d. Then $\overline{C}_2 = C_2 \cup D_{i+1}$ is a 2-cell in $G_i$. By Claim 4, $\overline{C}_2$ is good, and so, as in the above case, $G_i$ has a 3-cut $T$ such that $|T \cap \partial C_2| \geq 2$, and $T \cap \partial C_2 \subset \partial C_2 - \{u_1u_2, u_4u_5\}$. Hence $T$ is a 3-cut of $G_{i+1}$. Since $F_{i+1}$ contains the path $u'_7u_3u_2u'_5u'_1$, $F_{i+1}$ does not intersect $T$, a contradiction. See Figure 12e.

3. $p \leq i \leq p + q - 1$, i.e., $D_{i+1} \in D_2$.

Since $V(D_{i+1}) \subset F_{i+1}$, $|F_{i+1} \cap \partial D'_{i+1}|$ is 0, 2, 4 or 6.

**Case 1.** $|F_{i+1} \cap \partial D'_{i+1}| = 0$.

In this case, the 6-cycle $u_1u_2u_3u_4u_5u_7u_8$ is contained in $F_{i+1}$, and replacing it with the 8-cycle $u_1u_2u_3u_4u_5u_6u_7u_8$, we obtain the 8-cycle $F_i$ in $G_i$. Obviously, both A1 and A2 hold.

**Case 2.** $|F_{i+1} \cap \partial D'_{i+1}| = 2$.

Since $V(D_{i+1}) \subset F_{i+1}$ and $F_{i+1}$ is an even subgraph, by symmetry we have two cases. If $F_{i+1} \cap \partial D'_{i+1} = \{u_7u'_7, u_8u'_8\}$, then $F_{i+1}$ contains the path $u_8u_1u_3u_4u_5u_7$. See Figure 13a. Then the even subgraph $F_i$ obtained from $F_{i+1}$ by replacing $u'_8u_8u_1u_3u_4u_5u_7u'_7$ by $u'_8u_8u_1u_2u_3u_4u_5u_6u_7u'_7$ is a desired even subgraph because both of A1 and A2 hold. Similarly, we can show the case of $F_{i+1} \cap \partial D'_{i+1} = \{u_7u'_7, u_5u'_5\}$ since $u_6 \in R(D_{i+1}) \subset R_i$. See Figure 13b.

**Case 3.** $|F_{i+1} \cap \partial D'_{i+1}| = 4$.

By symmetry, we have four cases.

(i) $F_{i+1} \cap \partial D'_{i+1} = \{u_7u'_7, u_8u'_8, u_1u'_1, u_3u'_3\}$.

Since $V(D'_{i+1}) \subset F_{i+1}$, $F_{i+1}$ contains the paths $u_8u_1$ and $u_3u_4u_5u_7$. See Figure 14a. Let $P_1$ and $P_2$ be the two paths obtained from the cycles in $F_{i+1}$ intersecting $D'_{i+1}$.
by removing edges in $E(D'_{i+1})$ and isolated vertices. By symmetry, we may suppose $u_3 \in P_1$. If $u_7 \in P_1$ or $u_8 \in P_1$, then the even subgraph $F_i$ obtained from $F_{i+1}$ by replacing

$$u'_1u_1u_8u'_8 \text{ and } u'_3u_3u_4u_5u_7u'_7 \text{ by } u'_1u_1u_8u'_8 \text{ and } u'_3u_3u_4u_5u_6u_2u_1u'_1$$

is a desired even subgraph because both of A1 and A2 hold. See Figure 14a.

In the case of $u_1 \in P_1$, let $F_i$ be the even subgraph obtained from $F_{i+1}$ by replacing

$$u'_3u_3u_4u_5u_7u'_7 \text{ by } u'_3u_3u_4u_5u_6u_7u'_7.$$ 

Obviously A1 holds. Since $u_2 \in R(D_{i+1}) \subset R_i$, A2 holds.

(ii) $F_{i+1} \cap \partial D'_{i+1} = \{u_7u'_7, u_8u'_8, u_3u'_3, u_4u'_4\}$.

Then $F_{i+1}$ contains the paths $u_8u_1u_3$ and $u_4u_5u_7$. See Figure 14b. Then the even subgraph $F_i$ obtained from $F_{i+1}$ by replacing

$$u'_8u_8u_1u_3u'_3 \text{ and } u'_4u_4u_5u_7u'_7 \text{ by } u'_8u_8u_1u_2u_3u'_3 \text{ and } u'_4u_4u_5u_6u_7u'_7$$

is a desired even subgraph because both of A1 and A2 hold.

(iii) Similarly we can show the case that $F_{i+1} \cap \partial D'_{i+1} = \{u_7u'_7, u_8u'_8, u_4u'_4, u_5u'_5\}$. See Figure 14c.

(iv) $F_{i+1} \cap \partial D'_{i+1} = \{u_7u'_7, u_1u'_1, u_3u'_3, u_5u'_5\}$.

Then $F_{i+1}$ contains the paths $u_7u_8u_1$ and $u_3u_4u_5$. See Figure 14d.

Suppose that $u_4u_8 \in E(G)$. See Figure 15. Let $P_1$ and $P_2$ be the two paths obtained from the cycles in $F_{i+1}$ intersecting $D'_{i+1}$ by removing edges in $E(D'_{i+1})$. 

![Figure 14](image-url)
and isolated vertices. By symmetry, we may suppose $u_3 \in P_1$. If $u_1 \in P_1$ or $u_5 \in P_1$, then the even subgraph $F_i$ obtained from $F_{i+1}$ by replacing

$$u'_1u_1u_8u_7u'_7 \text{ and } u'_3u_3u_4u_5u'_5 \text{ by } u'_7u_7u_8u_4u_3u'_3 \text{ and } u'_1u_1u_2u_6u_5u'_5$$

is a desired even subgraph because both of A1 and A2 hold. See Figure 15i.

In the case of $u_7 \in P_1$, let $F_i$ be the even subgraph obtained from $F_{i+1}$ by replacing

$$u'_1u_1u_8u_7u'_7 \text{ and } u'_3u_3u_4u_5u'_5 \text{ by } u'_1u_1u_8u_4u_3u'_3 \text{ and } u'_7u_7u_6u_5u'_5.$$ 

See Figure 15ii. Obviously A1 holds. Since $u_2 \in R(D_{i+1}) \subset R_i$, A2 holds.

Therefore, we may assume that $u_4u_8 \notin E(G)$. Since there are components in $F_{i+1}$ containing $u_7u_8u_1$ and $u_3u_4u_5$, both of $u'_4$ and $u'_3$ exist. Thus $|E_{u_4,u_8}| = 2$, and so one of $u_4$ and $u_8$ is in $R_i$. By symmetry, we may suppose $u_4 \in R_i$. Let $F_i$ be the even subgraph obtained from $F_{i+1}$ by replacing

$$u'_3u_3u_4u_5u'_5 \text{ by } u'_3u_3u_2u_6u_5u'_5.$$ 

Obviously A2 holds.

Suppose the component $C_1$ of $F_i$ containing $u_7u_8u_1$ is a 5-cycle and $C_1 \cap S_i = \emptyset$. Let $C_1 = u_1u_8u_7u'_7u'_1u_4$ and $C_2 = u_1u_2u_6u_7u_8u_1$. See Figure 16a. Then $C = C_1 \cup C_2$
is a 2-cell in $G_i$. By Claim 2, $C$ is good in $G_i$. Since $D_{i+1}$ is bad, $C_2$ is bad, and so $C_1$ is good. Thus there is a 3-cut $T$ such that $|T \cap \partial C_1| \geq 2$. Since $C_2$ is bad, $T \cap \partial C_1 \subset \partial C_1 - \{u_1u_2, u_7u_6\}$. Hence $T$ is a 3-cut of $G_{i+1}$. Since $F_{i+1}$ contains $C_1$ as a component, $F_{i+1}$ does not intersect $T$, a contradiction. See Figure 16b.

Case 4. $|F_{i+1} \cap \partial D'_{i+1}| = 6$.
In this case, $F_{i+1}$ contains all the edges in $\partial D'_{i+1}$. Let $P_1$, $P_2$ and $P_3$ be the three paths obtained from the cycles in $F_{i+1}$ intersecting $D'_{i+1}$ by removing edges in $E(D'_{i+1})$ and isolated vertices. By symmetry, we may suppose $u_7 \in P_1$. It is easy to confirm that for all the following cases, both of A1 and A2 hold.

i). The ends of $P_1$ are $u_7$ and $u_8$.

a). If the ends of $P_2$ are $u_1$ and $u_3$, then let $F_i$ be the even subgraph obtained from $F_{i+1}$ by replacing the cycles in $F_{i+1}$ intersecting $D'_{i+1}$ with the cycle

$$u_7P_1u_8u_1P_2u_3u_4P_3u_5u_6u_7.$$

See Figure 17a.

b). If the ends of $P_2$ are $u_1$ and $u_4$, then let $F_i$ be the even subgraph obtained from $F_{i+1}$ by replacing the cycles in $F_{i+1}$ intersecting $D'_{i+1}$ with the cycle

$$u_7P_1u_8u_1P_2u_4u_3P_3u_5u_6u_7.$$
See Figure 17b.

c). If the ends of $P_2$ are $u_1$ and $u_5$, then let $F_i$ be the even subgraph obtained from $F_{i+1}$ by replacing the cycles in $F_{i+1}$ intersecting $D'_{i+1}$ with the cycle

$$u_7P_1u_8u_1P_2u_5u_4P_3u_3u_2u_6u_7.$$  

See Figure 17c.

Notice that by symmetry, we finished showing all the cases where there is a path joining $u_i$ and $u_{i+1}$ for any $i$ by the case i).

ii). The ends of $P_1$ are $u_7$ and $u_1$.

In this case, the ends of $P_2$ are $u_8$ and $u_4$; otherwise there is a path joining $u_i$ and $u_{i+1}$ for some $i$. Let $F_i$ be the even subgraph obtained from $F_{i+1}$ by replacing the cycles in $F_{i+1}$ intersecting $D'_{i+1}$ with the cycle

$$u_7P_1u_1u_8P_2u_4u_5P_3u_3u_2u_6u_7.$$  

See Figure 17d.

iii). The ends of $P_1$ are $u_7$ and $u_3$.

If the ends of $P_2$ are $u_8$ and $u_4$, then let $F_i$ be the even subgraph obtained from $F_{i+1}$ by replacing the cycles in $F_{i+1}$ intersecting $D'_{i+1}$ with the cycle

$$u_7P_1u_3u_2u_1P_3u_5u_4P_2u_8u_7.$$  

See Figure 17e.

If the ends of $P_2$ are $u_8$ and $u_5$, then let $F_i$ be the even subgraph obtained from $F_{i+1}$ by replacing the cycles in $F_{i+1}$ intersecting $D'_{i+1}$ with the cycle

$$u_7P_1u_3u_2u_1P_3u_4u_5P_2u_8u_7.$$  

See Figure 17f.

iv). The ends of $P_1$ are $u_7$ and $u_4$.

If the ends of $P_2$ are $u_8$ and $u_5$, then let $F_i$ be the even subgraph obtained from $F_{i+1}$ by replacing the cycles in $F_{i+1}$ intersecting $D'_{i+1}$ with the cycle

$$u_7P_1u_4u_5P_2u_8u_1P_3u_3u_2u_6u_7.$$  

23
See Figure 17g. The case that the ends of $P_2$ are $u_8$ and $u_3$ is same to the case iii). See Figure 17f.

v). The ends of $P_1$ are $u_7$ and $u_5$.

If the ends of $P_2$ are $u_8$ and $u_4$, then let $F_i$ be the even subgraph obtained from $F_{i+1}$ by replacing the cycles in $F_{i+1}$ intersecting $D'_{i+1}$ with the cycle $u_7P_1u_5u_4P_2u_8u_1P_3u_3u_2u_6u_7$.

See Figure 17h. The case that the ends of $P_2$ are $u_8$ and $u_3$ is same to the case iv). See Figure 17g.

Now we completes the proof. ■

4 Closing Remarks

The Travelling Salesman Problem (TSP) is the one to find a spanning closed walk of short length in a given graph. The typical method for TSP on 3-edge-connected cubic graphs is as follows; First, we find a 2-factor $F$ in a given 3-edge-connected cubic graph $G$, and take a certain connected graph $T$ (e.g. a spanning tree) in the graph $G/F$ obtained from $G$ by contracting all components in $F$, and then we obtain a connected subgraph $F \cup T$ of $G$. By modifying it suitably, we can get a spanning closed walk of length some function on $|E(T)|$. Since $T$ must be a connected subgraph of $G/F$, $|E(T)|$ is at least the number of components of $F$ minus one, and so the fewer number of components in $F$ gives the better bounds.

Aggarwal, Garg and Gupta [1] used Theorem A to start with a 2-factor having at most $n/5$ components, and showed the existence of a spanning closed walk of length at most $4n/3$ in a 3-edge-connected cubic graph of order $n$. This result was further improved to 2-edge connected or connected cubic graphs, graphs of maximum degree at most 3, or better bounds than $4n/3$, see [5, 8, 9].

Because of the above reasons, several researchers have been interested in a 2-factor in cubic graphs such that the number of 5-cycles are small, see [7]. Instead of using a 2-factor, we can use an even subgraph satisfying certain conditions on the order of each component. In fact, such structures have appeared in [8, 9] as intermediate products, which is called an $R$-factor in [8]. For those intermediate
products, it is not necessarily dominating, but the dominating property may help us to obtain good bounds, i.e., we expect that Theorem 1 has a potential application to problems on TSP.

References


[16] M.M. Matthews and D.P. Sumner, Hamiltonian results in $K_{1,3}$-free graphs, J. Graph Theory 8 (1984), 139-146.

