Long cycles in unbalanced bipartite graphs

Shuya Chiba\textsuperscript{1\dagger} Jun Fujisawa\textsuperscript{2\ddagger} Masao Tsugaki\textsuperscript{1\¶} Tomoki Yamashita\textsuperscript{3\||}

\textsuperscript{1}Department of Mathematical Information Science, Tokyo University of Science, 1-3 Kagurazaka, Shinjuku-ku, Tokyo 162-8601, Japan.

\textsuperscript{2}Faculty of Business and Commerce, Keio University, Yokohama, 223-8521 Japan.

\textsuperscript{3}Department of Mathematics, Kinki University 3-4-1 Kowakae, Higashi-Osaka, Osaka 577-8502, Japan.

Abstract

Let $G[X, Y]$ be a 2-connected bipartite graph with $|X| \geq |Y|$. For $S \subseteq V(G)$, we define $\delta(S;G) := \min\{d_G(v) : v \in S\}$. We define $\sigma_{1,1}(G) := \min\{d_G(x) + d_G(y) : x \in X, y \in Y, xy \notin E(G)\}$ and $\sigma_2(X) := \min\{d_G(x) + d_G(y) : x, y \in X, x \neq y\}$. We denote by $c(G)$ the length of a longest cycle in $G$. Jackson [J. Combin. Theory Ser. B 38 (1985), 118–131] proved that $c(G) \geq \min\{2\delta(X;G) + 2\delta(Y;G) - 2, 4\delta(X;G) - 4, 2|Y|\}$. In this paper, we extend this result, and prove that $c(G) \geq \min\{2\sigma_{1,1}(G) - 2, 2\sigma_2(X) - 4, 2|Y|\}$.

Keywords: Bipartite graph, Longest cycle, Degree

AMS Subject Classification: 05C38

\dagger Email address: chiba@rs.kagu.tus.ac.jp
\ddagger Supported by JSPS Grant-in-Aid for Young Scientists (B), 23740087, 2011
\¶ Email address: fujisawa@fbc.keio.ac.jp
\ddagger Supported by JSPS Grant-in-Aid for Young Scientists (B), 22740068, 2011
\¶ Email address: tsugaki@hotmail.com
\|| Email address: yamashita@math.kindai.ac.jp
1 Introduction

We consider only finite graphs without loops or multiple edges. For standard graph-theoretic terminology not explained in this paper, we refer the reader to [2]. The purpose of this paper is to investigate degree sum conditions for the existence of long cycles in unbalanced bipartite graphs. Let $G$ be a graph and $S \subseteq V(G)$. The length of a longest cycle in $G$ is denoted by $c(G)$. We denote by $d_G(v)$ and $G[S]$ the degree of a vertex $v$ in $G$ and the subgraph of $G$ induced by $S$, respectively. We define $\sigma_2(S) := \min\{d_G(u) + d_G(v) : u, v \in S, u \neq v, uv \notin E(G)\}$ if $G[S]$ is not complete; otherwise, let $\sigma_2(S) := +\infty$. We simply write $\sigma_2(G)$ instead of $\sigma_2(V(G))$. Concerning degree sum conditions for hamiltonicity of general graphs, the following is well-known.

Theorem 1 (Ore [8]) Let $G$ be a graph. If $\sigma_2(G) \geq |V(G)|$, then $G$ is Hamiltonian.

This theorem was extended as follows.

Theorem 2 (Bermond [1], Linial [6]) Let $G$ be a 2-connected graph. Then $c(G) \geq \min\{\sigma_2(G), |V(G)|\}$.

We next mention the bipartite graph versions of these theorems. We denote by $G[X,Y]$ a bipartite graph $G$ with partite sets $X$ and $Y$, and $G[X,Y]$ is called balanced if $|X| = |Y|$. For $G[X,Y]$, we define $\sigma_{1,1}(G) := \min\{d_G(x) + d_G(y) : x \in X, y \in Y, xy \notin E(G)\}$. Using this invariant, Moon and Moser [7] and Kaneko and Yoshimoto [5] gave sufficient conditions for Hamiltonicity and the existence of long cycles in balanced bipartite graphs, respectively.

Theorem 3 (Moon and Moser [7]) Let $G$ be a balanced bipartite graph. If $2\sigma_{1,1}(G) \geq |V(G)| + 2$, then $G$ is Hamiltonian.

Theorem 4 (Kaneko and Yoshimoto [5]) Let $G$ be a 2-connected balanced bipartite graph. Then $c(G) \geq \min\{2\sigma_{1,1}(G) - 2, |V(G)|\}$.

In this paper, we investigate the existence of long cycles in unbalanced bipartite graphs, which is not guaranteed by above theorems. As an unbalanced version of Theorem 3, we can obtain the following from [3, Corollary 3 and Lemma 8].
**Theorem 5** Let $G[X, Y]$ be a bipartite graph with $|X| \geq |Y|$. If $2\sigma_{1,1}(G) \geq |V(G)| + 2$, then either (i) $G$ is Hamiltonian or (ii) there exists a cycle in $G$ containing all vertices in $Y$.

On the other hand, there exists a 2-connected unbalanced bipartite graph $G$ such that $c(G) < \min\{2\sigma_{1,1}(G) - 2, 2|Y|\}$ (we will show it later). Hence we cannot obtain the unbalanced version of Theorem 4 by using these two invariants, and so a previous study in this line of research was done by also using the minimum degree of the larger partite set as follows, where for a graph $G$ and $S \subseteq V(G)$, we let $\delta(S; G) := \min\{d_G(v) : v \in S\}$.

**Theorem 6 (Jackson [4])** Let $G[X, Y]$ be a 2-connected bipartite graph with $|X| \geq |Y|$. Then $c(G) \geq \min\{2\delta(X; G) + 2\delta(Y; G) - 2, 4\delta(X; G) - 4, 2|Y|\}$.

The main result of this paper is the following, which is a generalization of Theorem 6 (note that by Theorem 4, Theorem 7 holds for the case $|X| = |Y|$).

**Theorem 7** Let $G[X, Y]$ be a 2-connected bipartite graph with $|X| > |Y|$. Then $c(G) \geq \min\{2\sigma_{1,1}(G) - 2, 2\sigma_2(X) - 4, 2|Y|\}$.

Here we make three remarks on Theorem 7.

(i) There exists a graph $G$ such that the lower bound of $c(G)$ which Theorem 7 guarantees is larger than Jackson’s theorem.

Let $k, l, m$ be integers with $l > k \geq 2$ and $m \geq k + l - 2$. For $i = 1, 2$, let $G_i[X_i, Y_i]$ be a complete bipartite graph with $|X_1| = 1$, $|X_2| = m$, $|Y_1| = k$ and $|Y_2| = l$. Let $G[X, Y]$ be a bipartite graph obtained from $G_1$ and $G_2$ by adding two vertices $x_1$ and $x_2$ and by joining each vertex in $\{x_1, x_2\}$ and each vertex in $Y_1 \cup Y_2$, and let $X := X_1 \cup X_2 \cup \{x_1, x_2\}$ and $Y := Y_1 \cup Y_2$. Then $G[X, Y]$ is a 2-connected bipartite graph such that $|X| > |Y|$, $\sigma_{1,1}(G) = l + 3$, $\sigma_2(X) = k + l$, $\delta(X; G) = k$ and $\delta(Y; G) = 3$. If $2 \leq k \leq 3$, then $\min\{2\delta(X; G) + 2\delta(Y; G) - 2, 4\delta(X; G) - 4, 2|Y|\} = 4k - 4 < 2(k + l) - 4 = \min\{2\sigma_{1,1}(G) - 2, 2\sigma_2(X) - 4, 2|Y|\}$; if $k \geq 4$, then $\min\{2\delta(X; G) + 2\delta(Y; G) - 2, 4\delta(X; G) - 4, 2|Y|\} = 2k + 4 < 2l + 4 = \min\{2\sigma_{1,1}(G) - 2, 2\sigma_2(X) - 4, 2|Y|\}$.
(ii) The invariant $\sigma_2(X)$ in Theorem 7 is necessary and sharp, i.e., there exists a 2-connected bipartite graph $G[X,Y]$ such that $\min\{2\sigma_{1,1}(G) - 2, 2|Y|\} > 2\sigma_2(X) - 4 = c(G)$.

Let $l, k$ be integers with $l \geq k \geq 3$. For $i = 1, 2, 3$, let $G_i[X_i, Y_i]$ be a complete bipartite graph with $|X_i| = l$ and $|Y_i| = k - 2$. Let $G[X,Y]$ be a bipartite graph obtained from $G_1, G_2$ and $G_3$ by adding two vertices $y_1$ and $y_2$ and by joining each vertex in $\{y_1, y_2\}$ and each vertex in $X_1 \cup X_2 \cup X_3$, and let $X := X_1 \cup X_2 \cup X_3$ and $Y := Y_1 \cup Y_2 \cup Y_3 \cup \{y_1, y_2\}$. Then $G[X,Y]$ is a 2-connected bipartite graph with $|X| > |Y|$, $\sigma_{1,1}(G) = k + l$, $\sigma_2(X) = 2k$, $|Y| = 3k - 4$ and $c(G) = 4k - 4$. Hence $\min\{2(\sigma_{1,1}(G) - 1), 2|Y|\} > 2\sigma_2(X) - 4 = c(G)$.

(iii) The invariant $\sigma_{1,1}(G)$ in Theorem 7 is necessary and sharp, i.e., there exists a graph $G[X,Y]$ such that $\min\{2\sigma_2(X) - 4, 2|Y|\} > 2\sigma_{1,1}(G) - 2 = c(G)$.

Let $n, m, l, k$ be integers with $n \geq k - 1 \geq l + 1 \geq 3$, $n + l \geq k + m + 1$ and $m \geq l$. For $i = 1, 2$, let $G_i[X_i, Y_i]$ be a complete bipartite graph with $|X_i| = n$, $|Y_i| = k$, $|X_2| = l$ and $|Y_2| = m$. Let $G[X,Y]$ be a graph obtained from $G_1$ and $G_2$ by joining each vertex in $Y_1$ and each vertex in $X_2$, and let $X := X_1 \cup X_2$ and $Y := Y_1 \cup Y_2$. Then $G[X,Y]$ is a 2-connected bipartite graph with $|X| > |Y|$, $\sigma_{1,1}(G) = k + l$, $\sigma_2(X) = 2k$ and $c(G) = 2(k + l - 1)$. Hence $\min\{2\sigma_2(X) - 4, 2|Y|\} > 2(\sigma_{1,1}(G) - 1) = c(G)$.

In the rest of this section, we prepare notation which we use in subsequent sections.

For a graph $G$, we denote by $N_G(v)$ the neighborhood of a vertex $v$ in $G$. Moreover, we denote $N_H(v) := N_G(v) \cap V(H)$ and $d_H(v) := |N_H(v)|$ for a subgraph $H$ of $G$, and let $N_H(S) := \bigcup_{v \in S} N_H(v)$ for a subset $S$ of $V(G)$. If there is no chance of confusion, we often identify a subgraph $H$ of $G$ with its vertex set $V(H)$. We write a cycle (or a path) $C$ with a given orientation by $\overrightarrow{C}$. If there is no chance of confusion, we abbreviate $\overrightarrow{C}$ by $C$. Let $\overrightarrow{C}$ be an oriented cycle or an oriented path. For $v \in V(C)$, we denote the $h$-th successor and the $h$-th predecessor of $v$ on $\overrightarrow{C}$ by $v^+_h$ and $v^-_h$, respectively. For $S \subseteq V(C)$ and a positive integer $h$, we define $S^{+h} := \{v^+_h : v \in S\}$. We abbreviate $v^{+1}$, $v^{-1}$ and $S^{+1}$ by $v^+$, $v^-$ and $S^+$, respectively. For $u, v \in V(C)$, we denote by $\overrightarrow{C}[u, v]$ and $\overrightarrow{C}(u, v)$ the path from $u$ to $v$ on $\overrightarrow{C}$ and the path $C[u, v] - \{u, v\}$ (possibly, $\overrightarrow{C}(u, v) = \emptyset$).
The reverse sequence of $\overrightarrow{C}[u, v]$ is denoted by $\overleftarrow{C}[v, u]$. A path with end vertices $u$ and $v$ is denoted by a $(u, v)$-path. For two subgraphs $H_1$ and $H_2$ of $G$, a $(u, v)$-path $P$ in $G$ is called an $(H_1, H_2)$-path if $V(P) \cap V(H_1) = \{u\}$ and $V(P) \cap V(H_2) = \{v\}$.

2 Lemmas

In the proof of Theorem 7, to estimate the length of a longest cycle $C$ of a graph $G$, we will use a long path with specified end vertices in $G - V(C)$. To see this, we prepare some lemmas in this section.

For a graph $G$, let $C(G)$ be the set of cut vertices of $G$ and $I(G) := V(G) \setminus C(G)$. Let $\mathcal{B}(G)$ be the set of blocks of a graph $G$. For $B \in \mathcal{B}(G)$, let $I(B; G) := V(B) \cap I(G)$ and $C(B; G) := V(B) \cap C(G)$.

**Lemma 1** Let $G$ be a connected graph. Then $|C(G)| = \sum_{B \in \mathcal{B}(G)} |C(B; G)| - |\mathcal{B}(G)| + 1$.

**Proof of Lemma 1.** Let $T$ be the block-cut tree of $G$, i.e., $T$ is a graph with the vertex set $\mathcal{B}(G) \cup C(G)$ and the edge set $\{Bc : B \in \mathcal{B}(G), c \in C(B; G)\}$. Then by the definition of a block-cut tree, $|V(T)| = |\mathcal{B}(G)| + |C(G)|$, $|E(T)| = \sum_{B \in \mathcal{B}(G)} |C(B; G)|$. Hence $|C(G)| = |V(T)| - |\mathcal{B}(G)| = (|E(T)| + 1) - |\mathcal{B}(G)| = \sum_{B \in \mathcal{B}(G)} |C(B; G)| - |\mathcal{B}(G)| + 1$. \hfill \Box

**Lemma 2** Let $G[X, Y]$ be a bipartite graph and let $W \subseteq X$. If $|X \setminus W| \geq |Y|$, then there exists $B_0 \in \mathcal{B}(G)$ such that $|I(B_0; G) \cap (X \setminus W)| \geq |V(B_0) \cap Y|$.

**Proof of Lemma 2.** If $G$ is 2-connected, then $G$ itself is the desired block. If $G$ is not connected, then there exists a component $H$ of $G$ such that $|(X \setminus W) \cap H| \geq |Y \cap H|$. Hence we may assume that the connectivity of $G$ is one. We define $\mathcal{B}(c) := \{B \in \mathcal{B}(G) : c \in B\}$ for each $c \in C(G)$. Then $\sum_{c \in C(G)} |\mathcal{B}(c)| = \sum_{B \in \mathcal{B}(G)} |C(B; G)|$. Moreover, by Lemma 1, $|C(G)| = \sum_{B \in \mathcal{B}(G)} |C(B; G)| - |\mathcal{B}(G)| + 1$. Thus we have

$$|C(G) \cap (X \setminus W)| + |C(G) \cap Y| \leq |C(G)|$$

$$= |\mathcal{B}(G)| - 1 - \sum_{c \in C(G)} |\mathcal{B}(c)| + 2|C(G)|$$

$$= |\mathcal{B}(G)| - 1 - \sum_{c \in C(G)} (|\mathcal{B}(c)| - 2). \quad (1)$$
On the other hand, by the definition of $B(c)$,

$$
\sum_{B \in B(G)} |V(B) \cap Y| = |Y| + \sum_{c \in C(G) \cap Y} (|B(c)| - 1) = |Y| + |C(G) \cap Y| + \sum_{c \in C(G) \cap Y} (|B(c)| - 2).
$$

(2)

Hence we obtain

$$
|C(G) \cap (X \setminus W)| + \sum_{B \in B(G)} |V(B) \cap Y| \leq |B(G)| - 1 - \sum_{c \in C(G) \cap X} (|B(c)| - 2) + |Y|
$$

by summing (1) and (2). Since $|C(G) \cap (X \setminus W)| = |X \setminus W| - \sum_{B \in B(G)} |I(B; G) \cap (X \setminus W)|$ and $|B(c)| - 2 \geq 0$ for every $c \in C(G)$, we have $\sum_{B \in B(G)} (|I(B; G) \cap (X \setminus W)| - |V(B) \cap Y| + 1) \geq 1$, that is, there exists $B_0 \in B(G)$ such that $|I(B_0; G) \cap (X \setminus W)| \geq |V(B_0) \cap Y|$. □

For a bipartite graph $G[X, Y]$ with $|X| \geq |Y|$ and $u, v \in V(G)$, we define

$$
\varepsilon(u, v; G) := \begin{cases} 
0 & \text{if } u \text{ and } v \text{ belong to different partite sets}, \\
1 & \text{if } u, v \in X \text{ and } |X| > |Y|, \\
-1 & \text{otherwise}.
\end{cases}
$$

Lemma 3 Let $G[X, Y]$ be a 2-connected bipartite graph. Let $u_1, u_2 \in V(G)$ with $u_1 \neq u_2$ such that either $u_1 \in X$ or $u_1, u_2 \in Y$. Let $W \subseteq X$ such that $|X \setminus W| \geq |Y|$. Then there exists a $(u_1, u_2)$-path of order at least $2\delta(X \setminus (W \cup \{u_1\}); G) + \varepsilon(u_1, u_2; G) \geq 2\delta(X \setminus W; G) + \varepsilon(u_1, u_2; G)$.

Proof of Lemma 3. We prove this lemma by induction on $|Y|$. If $|Y| = 2$, then $\delta(X \setminus (W \cup \{u_1\}); G) = 2$, and hence we can easily check that the conclusion holds.

First we consider the case $u_1 \in X$. We take $u'_1 \in N_G(u_1) \setminus \{u_2\}$ such that the number of components of $G - \{u_1, u'_1\}$ is as small as possible. Then the graph $G_0 := G - \{u_1, u'_1\}$ is connected since $G$ is 2-connected. Let $X_0 := X \setminus \{u_1\}$, $Y_0 := Y \setminus \{u'_1\}$ and $W_0 := W \setminus \{u_1\}$. Note that $|X_0 \setminus W_0| \geq |Y_0|$, since $|X \setminus W| \geq |Y|$. By Lemma 2, there exists $B \in B(G_0)$ such that $|I(B; G_0) \cap (X_0 \setminus W_0)| \geq |V(B) \cap Y_0|$. Let $X_B := V(B) \cap X_0$, $Y_B := V(B) \cap Y_0$ and $W_B := (V(B) \cap W_0) \cup (C(B; G_0) \cap X_0)$. Note that $|X_B \setminus W_B| = |I(B; G_0) \cap (X_0 \setminus W_0)| \geq |Y_B|$.}

6
Since \( G \) is 2-connected, we can take a \((u_1, B)\)-path \( \overrightarrow{P}_1 \) and a \((u_2, B)\)-path \( \overrightarrow{P}_2 \) which are vertex-disjoint. Note that \(|V(P_1)| \geq 2\) and \(|V(P_2)| \geq 1\). Choose \( P_1 \) and \( P_2 \) so that \(|V(P_1)| + |V(P_2)|\) is maximum. Let \( v_i \) be the end vertex of \( P_i \) in \( B \) for \( i = 1, 2 \). Since \(|X_B \setminus W_B| \geq |Y_B|\), it follows from the induction hypothesis that there exists a \((v_1, v_2)\)-path \( \overrightarrow{P} \) in \( B \) of order at least \( 2\delta(X_B \setminus (W_B \cup \{v^*\})); B \) + \( \varepsilon(v_1, v_2; B) \), where if \( v_2 \in X \), then \( v^* := v_2 \); otherwise \( v^* := v_1 \).

Case 1. \((N_G(u'_1) \cap V(B)) \setminus \{v_2\} = \emptyset\).

In this case \( d_B(v) = d_G(v) \) for all \( v \in X_B \setminus (W_B \cup \{v_2\}) \). Thus it follows that
\[
2\delta(X_B \setminus (W_B \cup \{v^*\})); B) + \varepsilon(v_1, v_2; B) \geq 2\delta(X \setminus (W \cup \{u_1\}); G) + \varepsilon(v_1, v_2; B).
\]
Therefore
\[
\overrightarrow{P}_1[u_1, v_1] \overrightarrow{P}[v_1, v_2] \overrightarrow{P}_2[v_2, u_2] \text{ is a } (u_1, u_2)-\text{path in } G \text{ of order at least } 2\delta(X \setminus (W \cup \{u_1\}); G) + \varepsilon(v_1, v_2; B) + |V(P_1)| + |V(P_2)| - 2.
\]
We show that
\[
\varepsilon(v_1, v_2; B) + |V(P_1)| + |V(P_2)| - 2 \geq \varepsilon(u_1, u_2; G).
\] (3)

If \(|V(P_1)| + |V(P_2)| \geq 4\), then clearly (3) holds. Thus we may assume that \(|V(P_1)| = 2\) and \(|V(P_2)| = 1\). Then \( v_1 \in Y \) and \( u_2 = v_2 \). If \( v_2 \in X \), then \( \varepsilon(v_1, v_2; B) = 0 \). If \( v_2 \in Y \), then \( \varepsilon(v_1, v_2; B) = -1 \) and \( \varepsilon(u_1, u_2; G) = 0 \). In both cases, (3) holds.

Case 2. \((N_G(u'_1) \cap V(B)) \setminus \{v_2\} \neq \emptyset\).

Since \( d_B(v) \geq d_G(v) - 1 \) for all \( v \in X_B \setminus W_B \), it follows that \( 2\delta(X_B \setminus (W_B \cup \{v^*\})); B) \geq 2(\delta(X \setminus (W \cup \{u_1\}); G) - 1) \). Therefore
\[
\overrightarrow{P}_1[u_1, v_1] \overrightarrow{P}[v_1, v_2] \overrightarrow{P}_2[v_2, u_2] \text{ is a } (u_1, u_2)-\text{path in } G \text{ of order at least } 2\delta(X \setminus (W \cup \{u_1\}); G) + \varepsilon(v_1, v_2; B) + |V(P_1)| + |V(P_2)| - 4.
\]
We show that
\[
\varepsilon(v_1, v_2; B) + |V(P_1)| + |V(P_2)| - 4 \geq \varepsilon(u_1, u_2; G).
\] (4)

Let \( v'_1 \in (N_G(u'_1) \cap V(B)) \setminus \{v_2\} \). Since \( u_1 u'_1 v'_1 \) is a \((u_1, B)\)-path, it follows from the choice of \( P_1 \) and \( P_2 \) that either \(|V(P_1)| \geq 3\) or \( u'_1 \in V(P_2) \) (and in this case, \(|V(P_2)| \geq 3\)). Hence \(|V(P_1)| + |V(P_2)| \geq 4\). In particular, if \( u'_1 \in V(P_2) \) or \( v_2 \neq u_2 \), then \(|V(P_1)| + |V(P_2)| \geq 5\). Suppose that \(|V(P_1)| + |V(P_2)| \geq 5\). Then we may assume that \( \varepsilon(v_1, v_2; B) = -1 \) and \( \varepsilon(u_1, u_2; G) = 1 \) (otherwise (4) holds). These imply that \( u_2 \in X \), and \( v_1, v_2 \in X \). Then \(|V(P_1)| + |V(P_2)| \) is even, and hence \(|V(P_1)| + |V(P_2)| \geq 6\). Thus, (4) holds. Hence we may assume that \(|V(P_1)| + |V(P_2)| \leq 4\), that is, \(|V(P_1)| = 3\) and \( V(P_2) = \{v_2\} = \{u_2\} \). Moreover we may assume that
implies that \( w_j \) end vertices since \( u \) is 2-connected, \( u \) belongs to different partite sets. By Lemma 3, there exists a \((u_1, B)\)-path \( P_1' \) with end vertices \( u_1, v_2' \) of order at least 4. Then \( |V(P_1')| + |V(P_2)| \geq 5 > |V(P_1)| + |V(P_2)| \). This contradicts the choice of \( P_1 \) and \( P_2 \), and thus the lemma holds in the case \( u_1 \in X \).

Next we consider the case \( u_1, u_2 \in Y \). Let \( G^* \) be a graph obtained from \( G \) by adding a new vertex \( u_1 \) and two edges \( u_1' u_1, u_1' u_2 \), and let \( X^* := X \cup \{u_1\}, Y^* := Y \) and \( W^* := W \). Note that \( G^* \) satisfies the assumption of Lemma 3, and \( \delta(X^* \setminus (W^* \cup \{u_1\}); G^*) = \delta(X \setminus (W \cup \{u_1\}); G) \). We already know that there exists a \((u_1, u_2)\)-path in \( G^* \) of order at least \( 2\delta(X^* \setminus (W^* \cup \{u_1\}); G^*) + \varepsilon(u_1, u_2; G^*) = 2\delta(X^* \setminus (W^* \cup \{u_1\}); G^*) + \varepsilon(u_1, u_2; G) \). Thus there exists a \((u_1, u_2)\)-path in \( G \) of order at least \( 2\delta(X \setminus (W \cup \{u_1\}); G) - 1 = 2\delta(X \setminus (W \cup \{u_1\}); G) + \varepsilon(u_1, u_2; G) \). This completes the proof of Lemma 3.

**Lemma 4** Let \( G[X, Y] \) be a connected bipartite graph with \( |X| \geq |Y| \) and \( u \in V(G) \). Then there exist a vertex \( v \in V(G) \) and a \((u, v)\)-path \( P \) in \( G \) of order at least \( 2\delta(X; G) \) such that \( N_G(v) \subseteq V(P) \).

**Proof of Lemma 4.** By Lemma 2, there exists \( B_0 \in B(G) \) such that \( |I(B_0; G) \cap X| \geq |V(B_0) \cap Y| \). Let \( X_0 := V(B_0) \cap X, Y_0 := V(B_0) \cap Y \) and \( W_0 := C(B_0; G) \cap X \). Note that \( |X_0 \setminus W_0| \geq |Y_0| \). Since \( G \) is connected, there exists a \((u_1, B_0)\)-path \( P_1 \). Let \( u_1 \) be the end vertex of \( P_1 \) in \( B_0 \) (possibly \( u_1 = u \)), and let \( u_2 \) be a vertex in \( B_0 \) such that \( u_1 \) and \( u_2 \) belong to different partite sets. By Lemma 3, there exists a \((u_1, u_2)\)-path \( P_2 \) of order at least \( 2\delta(X_0 \setminus W_0; B_0) + \varepsilon(u_1, u_2; B_0) \geq \varepsilon(X; G) \). We can easily see that there exist \( v \in V(G) \) and a \((u, v)\)-path \( P \) such that \( V(P_1) \cup V(P_2) \subseteq V(P) \) and \( N_G(v) \subseteq V(P) \). Since \( |V(P_1) \cup V(P_2)| \geq |V(P_2)| \geq 2\delta(X; G) \), the desired conclusion holds.

In the rest of this section, we prepare useful lemmas concerning the existence of long cycles in bipartite graphs.

For a path \( P \) of a graph \( G \) with end vertices \( u \) and \( v \), we call \( P \) a maximal path in \( G \) if \( N_G(u) \cup N_G(v) \subseteq V(P) \). The following lemma is proved by Jackson [4].
Lemma 5 (Jackson [4]) Let $G[X, Y]$ be a 2-connected bipartite graph with $|X| \geq |Y|$, and $P$ be a maximal path in $G$ whose end vertices are $u$ and $v$.

(i) If $u \in X$ and $v \in Y$, then $c(G) \geq \min\{|V(P)|, 2(d_G(u) + d_G(v) - 1)\}$.

(ii) If $u, v \in X$, then $c(G) \geq \min\{|V(P)| - 1, 2(d_G(u) + d_G(v) - 2)\}$.

The following lemma is proved by Broersma, Yoshimoto and Zhang [3], where for a cycle $C$ of a graph $G$, $C$ is called a dominating cycle of $G$ if $G - V(C)$ is edgeless. For the convenience of readers, we give the sketch of the proof of this lemma.

Lemma 6 (Broersma et al. [3]) Let $G[X, Y]$ be a connected bipartite graph with $|X| \geq |Y|$. If $G$ has a dominating cycle, then $c(G) \geq \min\{2\sigma_1(G) - 2, 2|Y|\}$.

Sketch of the proof of Lemma 6. Let $C$ be a longest dominating cycle with a given orientation. If $|V(C)| < 2|Y|$, then there exist $x \in X \setminus V(C)$ and $y \in Y \setminus V(C)$. Since $C$ is a longest dominating cycle, $|N_C(x)^+ \cap N_C(y)| \leq 1$ and $|N_C(y)^+ \cap N_C(x)| \leq 1$ holds. Therefore $|V(C)| \geq |N_C(x) \cup N_C(x)^+ \cup N_C(y) \cup N_C(y)^+| \geq 2|N_C(x)| + 2|N_C(y)| - 2 \geq 2\sigma_1(G) - 2$. □

3 Proof of Theorem 7

Let $C$ be a longest cycle in $G$ with a given orientation.

Claim 1 Let $H$ be a component of $G - V(C)$. Let $u$ and $v$ be two distinct vertices in $H$ such that $N_C(u) \neq \emptyset$, and let $P$ be a $(u, v)$-path in $H$ with $N_H(v) \subseteq V(P)$. If $v \in X$, then the theorem holds.

Proof. By the assumption of Claim 1, we can easily see that there exists a maximal path $Q$ in $G$ such that one of the end vertices is $v$ and $V(P \cup C) \subseteq V(Q)$. Suppose that $v \in X$. Let $w$ be an end vertex of $Q$ other than $v$. Since $C$ is a longest cycle and $|V(Q)| > |V(C)|$, we have $wv \notin E(G)$. Since $|V(Q)| \geq |V(C)|+2$, it follows from Lemma 5 and the maximality of $|V(C)|$ that $c(G) \geq 2(d_G(w) + d_G(v) - 1) \geq 2(\sigma_1(G) - 1)$ or $c(G) \geq 2(d_G(w) + d_G(v) - 2) \geq 2(\sigma_2(X) - 2)$ according as $w \in Y$ or $w \in X$, respectively. □
Since $|X| > |Y|$, there exists a component $H_0$ of $G - V(C)$ such that $|V(H_0) \cap X| > |V(H_0) \cap Y|$. Let $X_0 := V(H_0) \cap X$ and $Y_0 := V(H_0) \cap Y$. Since $G$ is 2-connected, it follows from Claim 1 that $|Y_0| \neq 1$. Choose $H_0$ so that $|Y_0| \geq 2$ if possible. We divide the proof into two cases depending on whether $|Y_0| = 0$ or $|Y_0| \geq 2$.

**Case 1.** $|Y_0| = 0$, that is, $H_0 \simeq K_1$.

By the choice of $H_0$, either $H \simeq K_1$ or $|V(H) \cap X| \leq |V(H) \cap Y|$ holds for every component $H$ of $G - V(C)$. If $H \simeq K_1$ for every component $H$ of $G - V(C)$, then $C$ is a dominating cycle of $G$, and hence Lemma 6 yields that $c(G) \geq \min\{2\sigma_1(G) - 2, 2|Y|\}$. Thus we may assume that there exists a component $H_1$ of $G - V(C)$ such that $|V(H_1) \cap X| \leq |V(H_1) \cap Y|$ and $H_1 \not\simeq K_1$. Let $\{x_0\} := X_0$. Since $G$ is 2-connected, there exist $y_0 \in N_C(x_0)$ and $w \in N_C(H_1)$ with $y_0 \neq w$. Choose $y_0$ and $w$ so that $|V(\overline{C}[y_0,w])|$ is as small as possible. Let $u \in N_{H_1}(w)$. Let $y_1 \in V(H_1) \cap Y$ be a vertex in $H_1$ such that $d_{H_1}(y_1) = \delta(V(H_1) \cap Y; H_1)$. By Lemma 4, there exists a vertex $v \in V(H_1)$ and a $(u,v)$-path $\overline{P}$ in $H_1$ such that $|V(P)| \geq 2\delta(V(H_1) \cap Y; H_1) = 2d_{H_1}(y_1)$ and $N_{H_1}(v) \subseteq V(P)$. By Claim 1, we may assume $v \in Y$.

By Lemma 5, we may assume that $N_C(x_0) \cap N_C(y_1)^+ = \emptyset$ (otherwise there exists a maximal $(x_0,v')$-path containing $V(C) \cup \{x_0,y_1\}$ for some $v' \in V(H_1)$, and hence by Lemma 5, the desired conclusion holds). Since $N_C(x_0) \cap N_C(y_1)^+ \subseteq Y \cap V(\overline{C}[w,y_0])$, it follows that $|V(\overline{C}[w,y_0])| \geq 2(d_G(x_0) + d_C(y_1)) - 1$. Then $x_0 \overline{C}[y_0,w] \overline{P}[u,v]$ is a maximal $(x_0,v)$-path of order at least $1 + \left(2(d_G(x_0) + d_C(y_1)) - 1\right) + 2d_{H_1}(y_1) \geq 2(d_G(x_0) + d_G(y_1) - 1).$ Hence by Lemma 5 (i), $|V(C)| \geq \min\{2(d_G(x_0) + d_G(y_1) - 1), 2(d_G(x_0) + d_C(v) - 1)\} \geq 2\sigma_1(G) - 2$.  

**Case 2.** $|Y_0| \geq 2$.

For $u,v \in V(H_0)$ with $u \neq v$, we call that $u$ and $v$ are normally linked if $u,v \in N_{H_0}(C)$ and $|N_C(u) \cup N_C(v)| \geq 2$.

**Claim 2** There exist $z_1,z_2 \in X_0$, $u_1,u_2 \in V(H_0)$ and a $(u_1,u_2)$-path $P_0$ in $H_0$ such that $z_1 \neq z_2$, $u_1 \neq u_2$, $u_1$ and $u_2$ are normally linked and one of the following holds:

(I) $|N_C(z_1) \cup N_C(z_2)| \leq 1$ and $|V(P_0)| \geq d_{H_0}(z_1) + d_{H_0}(z_2) - 1$.

(II) $|N_C(z_h)| \geq 2$, $|N_C(z_{3-h})| = 0$, $u_h = z_h$ for some $h = 1$ or $2$ and $|V(P_0)| \geq d_{H_0}(z_1) + d_{H_0}(z_2) - 1$. 

(III) $u_1 = z_1$, $u_2 = z_2$ and $|V(P_0)| \geq d_{H_0}(z_1) + d_{H_0}(z_2) + 1$.

Proof. Choose $z_1 \in X_0$ so that $d_{H_0}(z_1) = \delta(X_0; H_0)$. Since $|X_0| > |Y_0|$, we have $|X_0 \setminus \{z_1\}| \geq |Y_0|$. Hence by Lemma 2, there exists $B \in \mathcal{B}(H_0)$ such that $|I(B; H_0) \cap (X_0 \setminus \{z_1\})| \geq |V(B) \cap Y_0|$. Let $X_B := V(B) \cap X_0$, $Y_B := V(B) \cap Y_0$ and $W_B := (C(B; H_0) \cap X_B) \cup \{z_1\}$. Note that $|X_B \setminus W_B| = |I(B; H_0) \cap (X_0 \setminus \{z_1\})| \geq |Y_B|$. Choose $z_2 \in X_B \setminus W_B$ so that $d_B(z_2) = \delta(X_B \setminus W_B; B)$. Note that $z_1 \neq z_2$, $d_{H_0}(z_2) \geq d_{H_0}(z_1)$, $z_2 \in I(B; H_0)$ and $d_{H_0}(z_2) = d_B(z_2)$.

Since $G$ is 2-connected, there exist two vertex-disjoint $(B, C)$-paths $Q_1$ and $Q_2$. For $i = 1, 2$, let $v_i \in V(B)$ and $w_i \in V(C)$ be end vertices of $Q_i$, and let $u_i \in V(H_0)$ such that $u_i w_i \in E(Q_i)$. Choose $Q_1$ and $Q_2$ so that $|\{u_1\} \cap \{z_1\}| + |\{u_2\} \cap \{z_2\}|$ is as large as possible. Note that $u_1 \neq u_2$ and $u_1$ and $u_2$ are normally linked since $Q_1$ and $Q_2$ are vertex-disjoint paths. Let $P_i := Q_i - \{w_i\}$ for $i = 1, 2$.

We show that there exists a $(u_1, u_2)$-path in $H_0$ of order at least $d_{H_0}(z_1) + d_{H_0}(z_2) + \varepsilon(v_1, v_2; B) + |V(P_1)| + |V(P_2)| - 2$. If $B$ is 2-connected, then by Lemma 3, there exists a $(v_1, v_2)$-path in $B$ of order at least $2\delta(X_B \setminus W_B; B) + \varepsilon(v_1, v_2; B) \geq d_{H_0}(z_1) + d_{H_0}(z_2) + \varepsilon(v_1, v_2; B)$; if $B$ is not 2-connected, then $B$ must be $K_2$ and an end block, in particular, $V(B) = \{v_1, v_2\}$, $\varepsilon(v_1, v_2; B) = 0$ and $(d_{H_0}(z_1) \leq ) d_{H_0}(z_2) = 1$, and hence there exists a $(v_1, v_2)$-path in $B$ of order at least $d_{H_0}(z_1) + d_{H_0}(z_2) + \varepsilon(v_1, v_2; B)$. Therefore we obtain a $(u_1, u_2)$-path $P_0$ such that $|V(P_0)| \geq d_{H_0}(z_1) + d_{H_0}(z_2) + \varepsilon(v_1, v_2; B) + |V(P_1)| + |V(P_2)| - 2$. Note that $|V(P_0)| \geq d_{H_0}(z_1) + d_{H_0}(z_2) - 1$ since $\varepsilon(v_1, v_2; B) + |V(P_1)| + |V(P_2)| - 2 \geq -1$.

Suppose that $z_1$ and $z_2$ are not normally linked. If $|N_{C}(z_1) \cup N_{C}(z_2)| \leq 1$, then (I) holds. Otherwise $|N_{C}(z_h)| \geq 2$ and $|N_{C}(z_{3-h})| = 0$ for some $h = 1$ or 2. Then the choice of $Q_1$ and $Q_2$ implies that $u_h = z_h$, and thus (II) holds.

Suppose next that $z_1$ and $z_2$ are normally linked. Then the choice of $Q_1$ and $Q_2$ yields that $u_1 = z_1$ and $u_2 = v_2 = z_2$. Suppose first that $z_1 \in V(B)$. Then $u_1 = v_1 = z_1$ and $|X_B| \geq |X_B \setminus W_B| + |\{z_1\}| > |Y_B|$. Since $v_1, v_2 \in X$, we have $\varepsilon(v_1, v_2; B) = 1$. Therefore we obtain $|V(P_0)| \geq d_{H_0}(z_1) + d_{H_0}(z_2) + \varepsilon(v_1, v_2; B) + |V(P_1)| + |V(P_2)| - 2 = d_{H_0}(z_1) + d_{H_0}(z_2) + 1$. Suppose next that $z_1 \notin V(B)$. Then $|V(P_1)| \geq 2$. Furthermore, if $v_1 \in Y_B$, then $\varepsilon(v_1, v_2; B) = 0$; otherwise, $|V(P_1)| \geq 3$. Hence $|V(P_0)| \geq d_{H_0}(z_1) + d_{H_0}(z_2) + \varepsilon(v_1, v_2; B) + |V(P_1)| + |V(P_2)| - 2 \geq d_{H_0}(z_1) + d_{H_0}(z_2) + 1$. Thus (III) holds. □
We divide the proof of Case 2 into three cases according to which of (I)–(III) holds in Claim 2.

Subcase 2.1. (I) holds in Claim 2.

Since \( u_1 \) and \( u_2 \) are normally linked, we can take \( w_1 \in N_C(u_1) \) and \( w_2 \in N_C(u_2) \) with \( w_1 \neq w_2 \). Since \( C \) is a longest cycle, it follows from Claim 2 (I) that \(|V(C)| = |V(\overline{C}(w_1, w_2))| + |V(\overline{C}(w_2, w_1))| + |\{w_1\}| + |\{w_2\}| \geq 2(d_{H_0}(z_1) + d_{H_0}(z_2) - 1) + 2. \) Since \(|N_C(z_1) \cup N_C(z_2)| \leq 1, |V(C)| \geq 2(d_C(z_1) + d_C(z_2) - 3) + 2 \geq 2\sigma_2(X) - 4. \)

Subcase 2.2. (II) holds in Claim 2.

In this case, there exist \( w_1 \in N_C(u_{3-h}) \) and \( w_2, w_3 \in N_C(z_h) \) with \( w_1 \neq w_2 \) and \( w_1 \neq w_3 \) (possibly \( w_2 = w_3 \)) such that \( w_1, w_2, w_3 \) are arranged in this order along \( \overline{C}, V(\overline{C}(w_1, w_2)) \cap N_C(z_h) = \emptyset \) and \( V(\overline{C}(w_3, w_1)) \cap N_C(z_h) = \emptyset. \) Since \( N_C(z_h) \setminus \{w_1\} \subseteq Y \cap V(\overline{C}[w_2, w_3]), \) we have \(|V(\overline{C}[w_2, w_3])| \geq 2(d_C(z_h) - 1) - 1 = 2d_C(z_h) - 3. \) Since \( C \) is a longest cycle, it follows from Claim 2 (II) that \(|V(\overline{C}(w_1, w_2))| + |V(\overline{C}(w_3, w_1))| \geq 2(d_{H_0}(z_h) + d_{H_0}(z_{3-h}) - 1) = 2(d_{H_0}(z_h) + d_C(z_{3-h}) - 1), \) and hence \(|V(C)| = |V(\overline{C}(w_1, w_2))| + |V(\overline{C}(w_3, w_1))| + |V(\overline{C}[w_2, w_3])| + |\{w_1\}| \geq 2(d_{H_0}(z_h) + d_C(z_{3-h}) - 1) + (2d_C(z_3) - 3) + 1 = 2(d_C(z_1) + d_C(z_2) - 2) \geq 2\sigma_2(X) - 4. \)

Subcase 2.3. (III) holds in Claim 2.

In this case, there exist \( w_1, w_2, w_3, w_4 \in N_C(z_1) \cup N_C(z_2) \) such that \( w_1, w_2, w_3, w_4 \) are arranged in this order along \( \overline{C}, w_1 \neq w_2, w_3 \neq w_4, z_1, z_2 \in N_{H_0}(\{w_1, w_2\}) \cap N_{H_0}(\{w_3, w_4\}), (V(\overline{C}(w_1, w_2)) \cup V(\overline{C}(w_3, w_4))) \cap N_C(\{z_1, z_2\}) = \emptyset \) and \( V(\overline{C}(w_1, w_2)) \cap V(\overline{C}(w_3, w_4)) = \emptyset. \) Let \( C_1 := \overline{C}(w_1, w_2), C_2 := \overline{C}[w_2, w_3], C_3 := \overline{C}(w_3, w_4) \) and \( C_4 := \overline{C}[w_3, w_4]. \) Since \( C \) is a longest cycle, it follows from Claim 2 (III) that \(|V(C_1)| + |V(C_3)| \geq 2(d_{H_0}(z_1) + d_{H_0}(z_2) + 1). \) Note that \( N_C(z_1) \cap N_C(z_2) = \emptyset \) because \( C \) is longest. Note also that \( N_C(z_1) \cup N_C(z_2)^{+2} \subseteq (Y \cap (V(C_2) \cup V(C_4))) \cup \{w_1^{+2}, w_3^{+2}\}. \) Hence \(|V(C_2)| + |V(C_4)| \geq 2(2(d_C(z_1) + d_C(z_2) - 1) - 1) + 2(d_C(z_1) + d_C(z_2) - 1) - 1 \geq 2(d_C(z_1) + d_C(z_2) - 3). \) Thus we obtain \(|V(C)| = |V(C_1)| + |V(C_3)| + |V(C_2)| + |V(C_4)| \geq 2(d_{H_0}(z_1) + d_{H_0}(z_2) + 1) + 2(d_C(z_1) + d_C(z_2) - 3) = 2(d_C(z_1) + d_C(z_2) - 2) \geq 2\sigma_2(X) - 4. \)

This completes the proof of Theorem 7. \( \square \)
References


